Folgen und Reihen

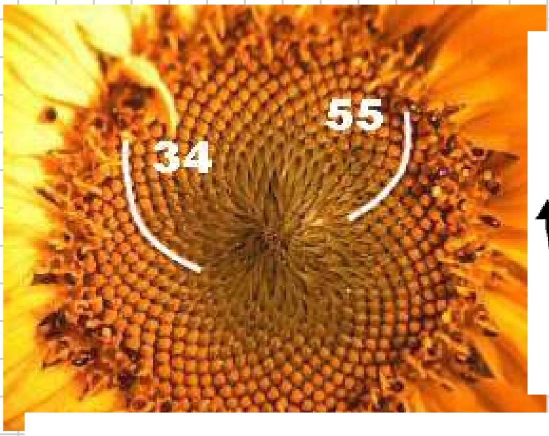
Denksport

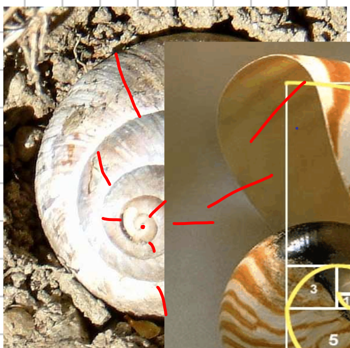
Setze folgende (Zahlen)-Folgen fort

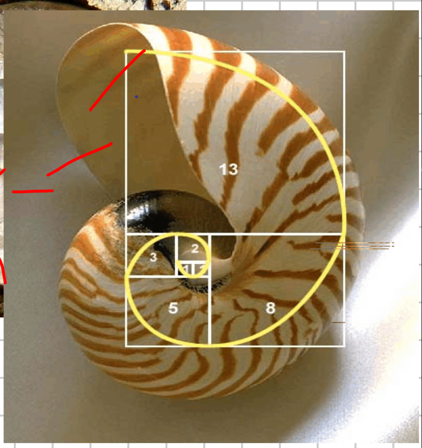
- 1 2 4 8 16 ...
- 1 1 2 3 5 8 ...
- E Z D V F 6 ...
- 1 4 9 16 25 ...
- 1 0 2 -1 3 -2 ...

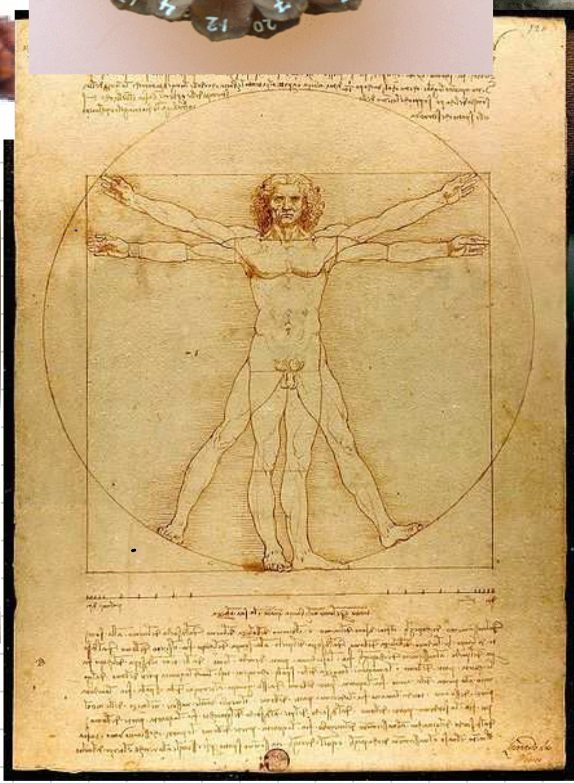
→ Fibonacci-Folge

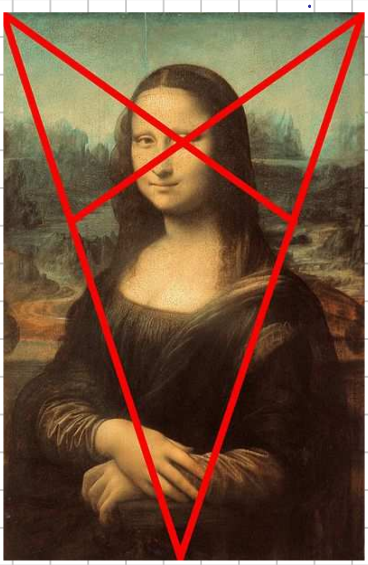
(* um 1180 in Pisa; † nach 1241 in Pisa)











→ Etwas Theorie

FOLGEN

S. 7/1

$n \rightarrow a_n$ $\qquad \langle a_n \rangle$ Folge

Natürliche Z → Folgenglied

n	a_n
1	1
2	1
3	2
4	3
5	5
⋮	⋮

Bsp. Unendliche Folge $\langle a_n \rangle = \langle 1; \frac{1}{2}; \frac{1}{3}; \frac{1}{4}; \frac{1}{5}; \dots \rangle$ aufzähl. Verf.

$\langle a_n \rangle = \langle \frac{1}{n} \rangle$ $\quad n \in \mathbb{N}$ Bildungsgesetz

Bsp $\langle 2, 3, 5, 7, 11, 13, 17, 19, 23, \dots \rangle$ Primzahlfolge

Bsp. $\langle a_n \rangle = \left\langle \frac{3n+1}{n+2} \right\rangle$

$\left[\left[f(x) = \frac{3x+1}{x+2} \right]\right]$

TI82 → TABLE

Explizite D.

$a_1 = \frac{3 \cdot 1 + 1}{1+2} = \frac{4}{3} = 1.333$

$a_2 = \frac{7}{4} = 1,75$

$a_3 = \frac{10}{5} = 2$

$a_4 = \frac{13}{6} = 2,1\dot{6}$

⋮

$a_{100.000} = \frac{300.001}{100.002} = 2,99995\dots$

Bsp Fibonacci - Folge

$a_0 = 1 \qquad a_1 = 1 \qquad a_2 = 2 \qquad a_3 = 3$

$\dots a_{100.000} = a_{99.999} + a_{99.998}$

Rekursive Darst.

$a_{n+1} = a_n + a_{n-1}$

Bsp. Rekursive Folge

$$\boxed{a_n = a_{n-1} \cdot 2 \qquad a_1 := 1}$$

$$a_2 = 1 \cdot 2 = 2$$
$$a_3 = 2 \cdot 2 = 4$$
$$a_4 = 4 \cdot 2 = 8$$
$$\vdots$$
$$\langle a_n \rangle = 2^{n-1} \qquad n = 1, 2, 3, 4, \dots$$
$$a_2 = 2^{2-1} = 2^1 = 2$$

Monotonie

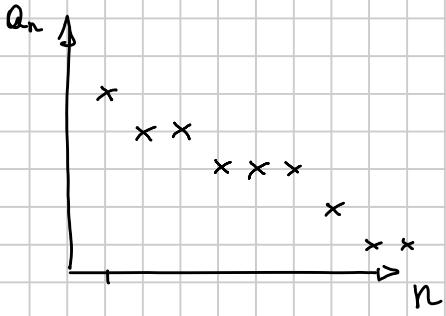
Bsp. ∘ $a_n = \frac{1}{n}$ $\frac{1}{1}; \frac{1}{2}; \frac{1}{3}; \frac{1}{4}; \frac{1}{5}; \dots$

Folgenglieder werden immer kleiner

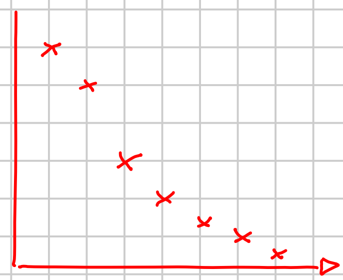
⇒ Folge ist monoton fallend (streng)

$$a_{n+1} < a_n$$

• $\langle 5, 4, 4, 3, 3, 3, 2, 1, 1, 1, 1, 0 \rangle$ monoton fallend



monoton fallend



streng monot. fall.

• Bsp. streng monoton wachsende Folge

$$\langle a_n \rangle = \langle n^2 \rangle = \langle 1, 4, 9, 16, 25, \dots \rangle$$

Bsp. 1. $\underline{\underline{\langle a_n \rangle}} = \underline{\underline{\left\langle \frac{n^2}{2n+3} \right\rangle}} = \left\langle \frac{1}{5}; \frac{4}{7}; \frac{9}{9}; \frac{16}{11}; \frac{25}{13}; \dots \right\rangle$

$0{,}2 < 0{,}\dots < 1 < 1{,}\dots < 1{,}9 < \dots$

<u>Vermutung:</u> streng monoton wachsend ✓

$a_{n+1} > a_n$

Google ~ Googol 10^{100}

$\frac{(n+1)^2}{2\cdot(n+1)+3} > \frac{n^2}{2n+3}$

$a_{Franz} = \frac{Franz^2}{2\,Franz+3}$

$\frac{1}{2} > \frac{1}{3} \;|\cdot 2 \;|\cdot 3$

$3 > 2$

$\frac{n^2+2n+1}{2n+\cancel{2+3}_{5}} > \frac{n^2}{2n+3} \quad |\cdot(2n+3) \;|\cdot(2n+5)$

$(n^2+2n+1)\cdot(2n+3) > n^2(2n+5)$

$2n^3+3n^2+4n^2+6n+2n+3 > 2n^3+5n^2$

$\cancel{2n^3}+7n^2+8n+3 > \cancel{2n^3}+5n^2 \quad | -5n^2$

$\underbrace{\underbrace{2n^2}_{>0}+\underbrace{8n}_{>0}+\underbrace{3}_{>0}}_{>0} > 0 \qquad \underline{\underline{n \in \mathbb{N}}}$

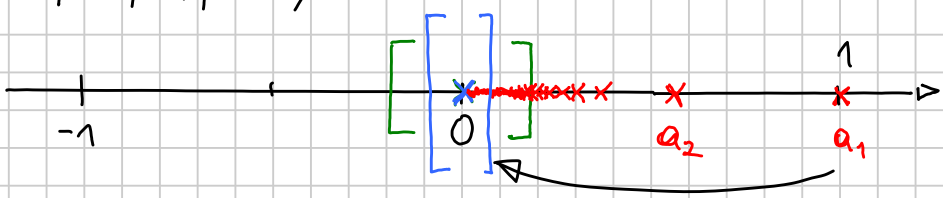
Grenzwert
(Limes)

S. 7/10

a_n Folge

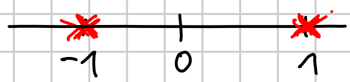
$\forall \varepsilon > 0 : \exists \underset{8}{n_0} \in \mathbb{N} : \forall \underset{9,10,11,\dots}{n} > n_0 : \; |a_n - a| < \varepsilon$ ∞

Bsp. $\left\langle \frac{1}{n} \right\rangle = \left\langle 1; \frac{1}{2}; \frac{1}{3}; \frac{1}{4}; \dots \right\rangle$



$\lim\limits_{n\to\infty} \frac{1}{n} = 0$

Bsp. $\langle -1; +1; -1; +1; -1; \dots \rangle$



Bsp. $\left\langle \left(1+\frac{1}{n}\right)^n \right\rangle \to e$ $\qquad 1^{1000} \dots 1$

Bsp.: • $\left\langle \frac{3}{n^2} \right\rangle$ $\qquad \lim\limits_{n\to\infty} \frac{3}{n^2} = 0$ $\qquad$ konvergent

• $\lim\limits_{n\to\infty} \frac{n}{2} = \infty$ $\qquad$ divergent

• $\lim\limits_{n\to\infty} 2^n = \infty$

$0{,}1 \quad 0{,}1\cdot 0{,}1 \quad 0{,}1\cdot 0{,}1\cdot 0{,}1$

$0{,}1 \;|\; 0{,}01 \;|\; 0{,}001$

• $\lim\limits_{n\to\infty} 0{,}1^n = 0$

• $\lim\limits_{n\to\infty} \frac{3n+1}{n+2} = \lim\limits_{n\to\infty} \frac{\cancel{n}\cdot\left(3+\frac{1}{n}\right)}{\cancel{n}\cdot\left(1+\frac{2}{n}\right)} = \frac{\lim\limits_{n\to\infty}\left(\underbrace{3}_{\to 3}+\underbrace{\frac{1}{n}}_{\to 0}\right)}{\lim\limits_{n\to\infty}\left(\underbrace{1}_{\to 1}+\underbrace{\frac{2}{n}}_{\to 0}\right)} =$

$\frac{3}{1} = \underline{\underline{3}}$

• $\lim\limits_{n\to\infty} \frac{4n-7}{2n+1} = \frac{4}{2} = 2$

• $\lim\limits_{n\to\infty} \frac{\boxed{7}n^2+2n-1}{\boxed{14}n^2-8n+7554} = \frac{7}{14} = \frac{1}{2}$ $\qquad \frac{2\cancel{n}}{n^{\cancel{2}}} = \frac{2}{n} \to 0$

• $\lim\limits_{n\to\infty} \frac{n^2+3}{n} = \infty$

• $\lim\limits_{n\to\infty} \frac{n+4}{n^3+1} = 0$

Zählergrad $\quad$ ZG

Nennergrad $\quad$ NG

ZG = NG $\to \frac{a}{b}$ $\quad$ Leitkoeff.

ZG > NG $\to \pm\infty$

ZG < NG $\to 0$

• $\lim\limits_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e = 2{,}71828\ldots$ $\quad$ Euler'sche Zahl

S. 7/10 $\quad \lim\limits_{n\to\infty} \frac{n}{3n-1} = \frac{1}{3}$

S. 7/13

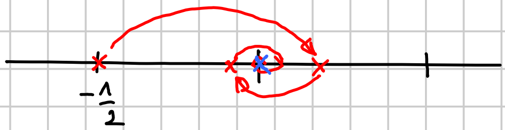
1.1. $\lim\limits_{n\to\infty} 3-2\cdot n = -\infty$

1.2. $\boxed{1}$ $\quad \frac{1}{2}; \frac{2}{3}; \frac{3}{4}; \frac{4}{5}; \ldots$

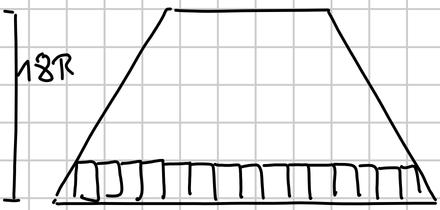
1.5. $\lim\limits_{n\to\infty} \frac{2^n}{2n+1} = \infty$ $\quad$ Exp $\gg$ Potenz $\gg$ Log

1.6. $\lim\limits_{n\to\infty} \frac{(-1)^n}{3n-1}$

$$\frac{-1}{2}; \frac{+1}{5}; \frac{-1}{8}; +\frac{1}{11}; \ldots$$



Arithmetische Folge



Ziegeldach

$a_1 = \underline{\underline{150}}$ Ziegel
$\quad a_1$

$a_2 = 145$ Ziegel $\quad \underline{\underline{-5 \text{ Z.}}}$
$\quad d$

$\vdots$

$a_{18} = 150 - 5\cdot(18-1)$

$\vdots$

$$\underline{\underline{a_n = a_1 + d\cdot(n-1)}}$$

Arithmetische Folge
$a_{n+1} - a_n = d$

Bsp. $\quad$ Sektglas-Pyramide

$a_1 = 60$ (1)

$a_5 = 1$ $\quad$ Höhe: 5 Etagen

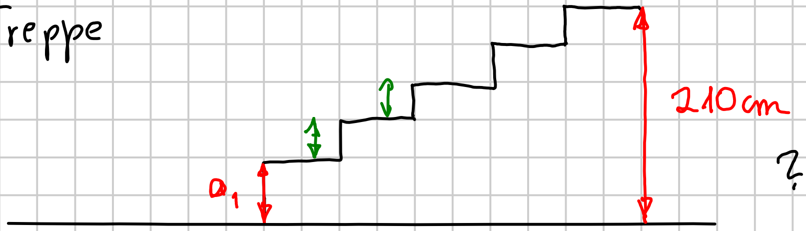
$a_5 = a_1 + (n-1)\cdot \underline{\underline{d}}$

$1 = 60$ (1) $+ 4\cdot d \quad | -60$

-59 (60) $= 4d \quad |:4$

$-15 = -\frac{60}{4} \quad -\frac{59}{4} = d$

Bsp. Treppe



? Wie viele Stufen muss ich steigen?

$a_1 = 30\,cm$ 1. Stufe

$d = 18\,cm$

$$a_n = a_1 + (n-1)\cdot d$$
$$210 = 30 + (n-1)\cdot 18 \quad \Big|$$
$$180 = (n-1)\cdot 18 \quad \Big| :18$$
$$10 = n-1 \quad \Big| +1$$
$$\underline{11 = n} \qquad \Rightarrow \quad 11 \text{ Stufen}$$

Geometrische Folge

Bsp. Schachbrett + Reiskörner

$$b_1 = 1 \xrightarrow{\cdot 2} b_2 = 2 \xrightarrow{\cdot 2} b_3 = 4$$
$\vdots$

$$b_1 = b_1$$
$$b_2 = b_1 \cdot q \qquad (\cdot q)$$
$$b_3 = b_1 \cdot q \cdot q = b_1 \cdot q^2 \qquad (\cdot q)$$
$$b_4 = b_1 \cdot q^3 \qquad (\cdot q)$$
$\vdots$
$$b_n = b_1 \cdot q^{n-1}$$

$$\frac{b_2}{b_1} = q$$

Bsp. Gerüchteküche

$$b_1 = 1 \xrightarrow{\cdot 3} b_2 = 3 \xrightarrow{\cdot 3} b_3 = 9$$
$\vdots$
$$b_n = 1 \cdot 3^{n-1}$$

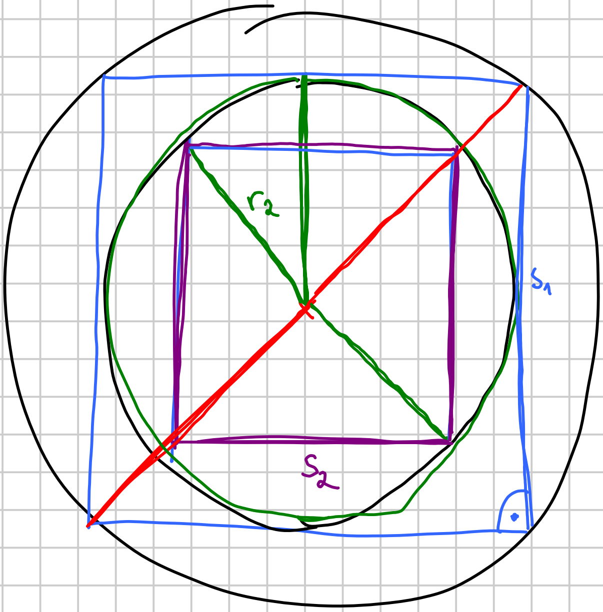
$q = 3$

[Wachstumsfkt. $N(t) = N_0 \cdot a^t$]

Bsp a) $b_n = 1000 \Rightarrow n = ?$

Bsp b) $n = 10 \Rightarrow b_{10} = 3^{10}$

$= \underline{\underline{59049}}$



Kreis $r_1 = 10\,cm$

? Umfang $u_1 =$

$$s_1^2 + s_1^2 = (2r_1)^2$$
$$2s_1^2 = 4r_1^2$$
$$s_1^2 = 2r_1^2$$
$$s_1 = \sqrt{2r_1^2} = \sqrt{2}\cdot r_1 = 14{,}14\,cm$$
$$u_1 = 4\cdot s_1 = 4\sqrt{2}\cdot r_1 = 56{,}5\,cm$$

$\sqrt{x^2} = x.$

$$? \; r_2 = \frac{s_1}{2} = \frac{\sqrt{2}\cdot r_1}{2}$$
$$r_2 = r_1 \cdot \boxed{\frac{\sqrt{2}}{2}} \; [q]$$

$$s_2^2 + s_2^2 = (2r_2)^2$$
$$\vdots$$
$$s_2 = \sqrt{2}\cdot r_2$$

$$r_n = r_1 \cdot q^{n-1}$$
$$r_n = 10\cdot\left(\frac{\sqrt{2}}{2}\right)^{n-1}$$
$$\Rightarrow \quad r_3 = 10\cdot\left(\frac{\sqrt{2}}{2}\right)^2$$
$$r_3 = 10\cdot\frac{\cancel{2}\,1}{\cancel{4}\,2} = 5$$

⋮

REIHEN

S. 7/16

Bsp: $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\dots =$

$$= \sum_{k=1}^{n} \frac{1}{k} = \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{n}$$

Summenschreibweise

Bsp: $\sum_{k=10}^{15}(k-3) = 7+8+9+10+11+12$

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\dots \to \infty$$

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\dots \to \underline{\underline{\frac{\pi^2}{6}}}$$

$$1+\frac{1}{1}+\frac{1}{2\cdot1}+\frac{1}{3\cdot2\cdot1}+\frac{1}{4\cdot3\cdot2\cdot1}+\dots = e$$

sin 30° $\left(\frac{30}{2\pi}\right)\frac{1}{2\cdot1}+\frac{1}{4\cdot3\cdot2\cdot1}+\frac{1}{6\cdot5\cdot4\cdot3\cdot2\cdot1}$