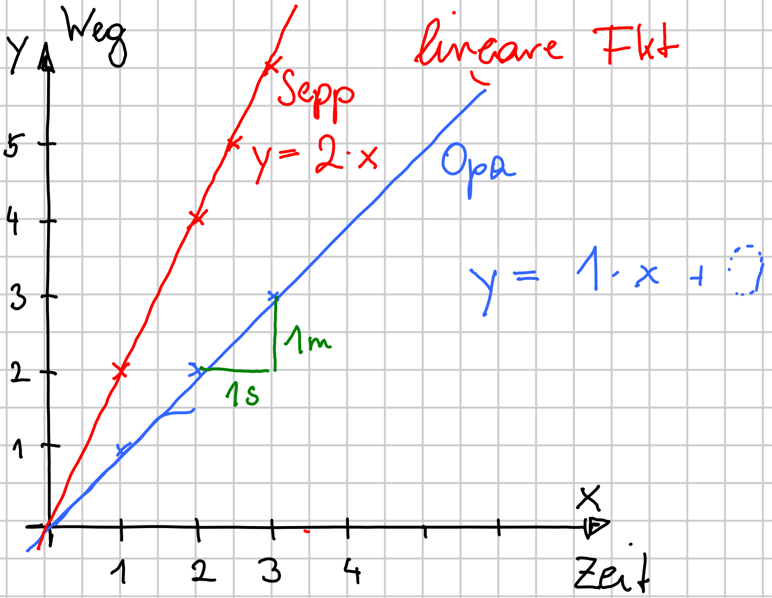
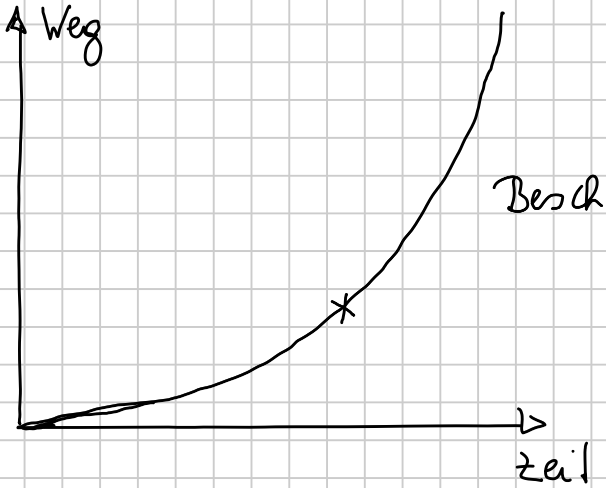
Differentialrechnung

Bsp. Sepp. - Spaziergang

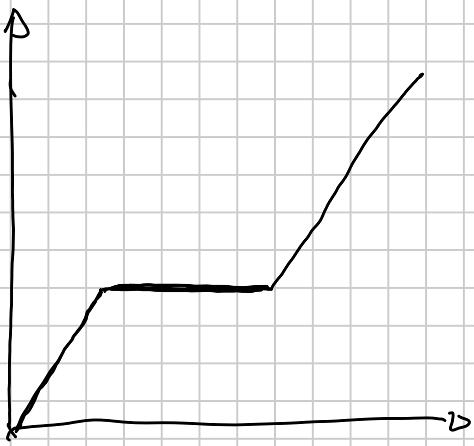
Zeit (t)	Weg (s)	Weg (s)
1	2 m	1
2	4 m	2
3	6 m	3
4	8 m	4
5	10 m	⋮
6	12 m	⋮

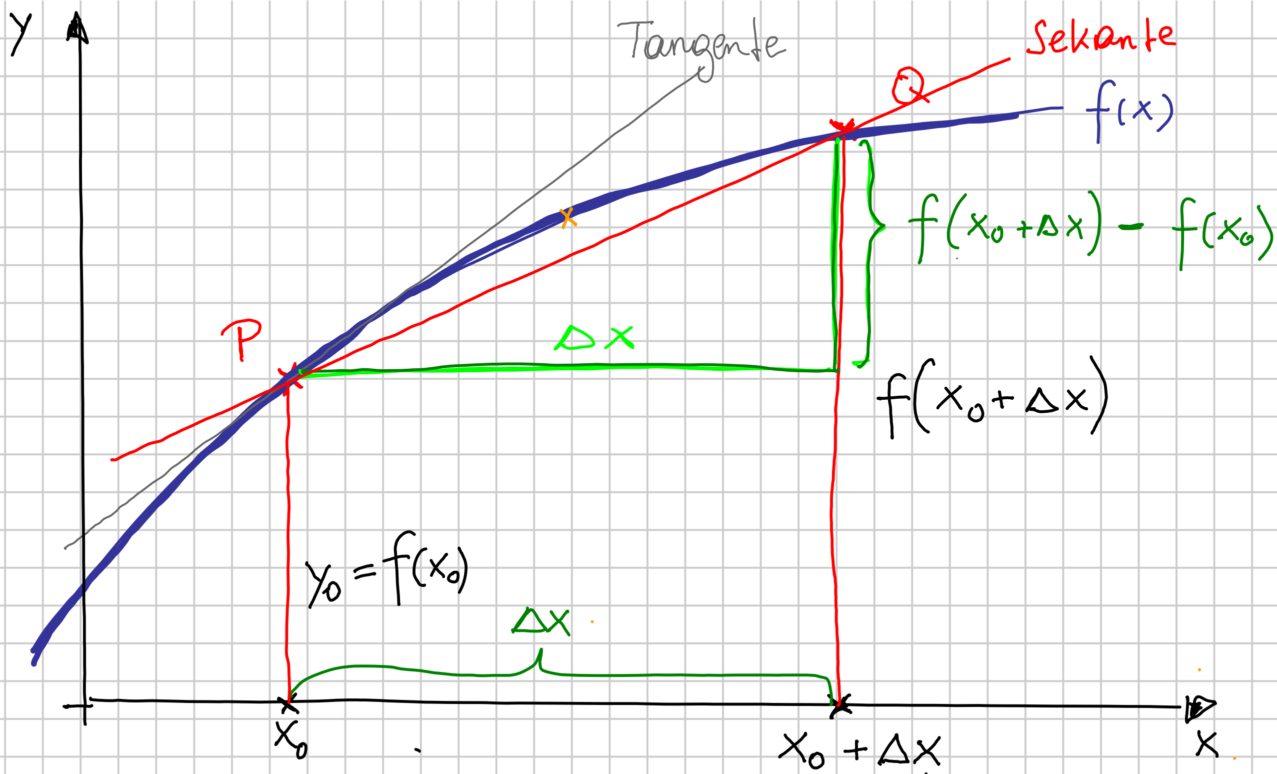


Geschwindigk.: $V = \frac{s}{t} = \frac{1m}{1s} = 1\,m/s \qquad \frac{km}{h}$



Beschleunigung





Tangentenproblem   Pr.: Momentangeschwindigkeit

$\left.\begin{matrix} P(3|4) \\ Q(6|7) \end{matrix}\right\}$ ⇒ Steigung Sekante $k_s = \frac{\Delta y}{\Delta x} = \frac{7-4}{6-3} = \frac{3}{3} = 1$

$$k_s = \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x} =$$ Sekantensteigung Differenzenquotient

$$\lim_{\Delta x \to 0} \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x}$$ Differentialquotient

Bsp. $f(x) = x^2$ Steigung der Tangente im Punkt

$f(\text{Franz}) = \text{Franz}^2$ $P(2|4)$? $x_0 = 2$

$$k_t = \lim_{\Delta x \to 0} \cancel{\frac{f(x+\Delta x) - f(x)}{\Delta x}} = \cancel{\lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x}} =$$

$$\lim_{\Delta x \to 0} \frac{\cancel{x^2} + 2x \cdot \Delta x + (\Delta x)^2 - \cancel{x^2}}{\Delta x} = \lim_{\Delta x \to 0} \frac{2x \cdot \Delta x + (\Delta x)^2}{\Delta x} =$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{\Delta x}(2x + \Delta x)}{\cancel{\Delta x}} = \lim_{\Delta x \to 0} 2 \cdot x + \underbrace{\Delta x}_{\to 0} = \underline{\underline{2x}}$$

k an $x = 1$? — $k = 2 \cdot 1 = 2$

$\Rightarrow$ Steigungsfunktion $f'(x) = 2 \cdot x$ = 1. Ableitung

Ableitungen

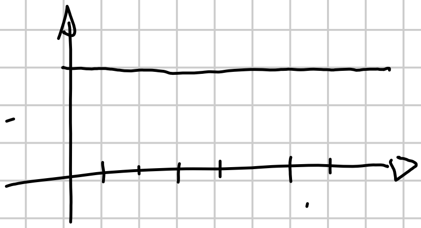
$f(x) = x^2$ $\qquad f'(x) = 2 \cdot x$

$f(x) = x^3$ $\qquad f'(x) = 3 \cdot x^2$

$f(x) = x^4$ $\qquad f'(x) = 4 \cdot x^3$

$f(x) = 3$

$f'(x) = 0$



$f(x) = x^{27} \quad \to \quad f'(x) = 27 \cdot x^{26}$

$f(x) = x^{-3} = \frac{1}{x^3} \quad \to \quad f'(x) = -3 \cdot x^{-4} = -3 \cdot \frac{1}{x^4}$

$f(x) = \frac{1}{x^7} = x^{-7} \quad \to \quad f'(x) = -7 \cdot x^{-8} = -7 \cdot \frac{1}{x^8} = -\frac{7}{x^8}$

$f(x) = x^{\frac{5}{2}} = \sqrt[2]{x^5} \quad \to \quad f'(x) = \frac{5}{2} \cdot x^{\frac{3}{2}} = \frac{5}{2} \cdot \sqrt[2]{x^3}$

$f(x) = \sqrt[2]{x} = x^{\frac{1}{2}} \quad \to \quad f'(x) = \frac{1}{2} \cdot x^{-\frac{1}{2}} = \frac{1}{2} \cdot \frac{1}{x^{\frac{1}{2}}} = \frac{1}{2} \cdot \frac{1}{\sqrt[2]{x^1}}$

$= \frac{1}{2 \cdot \sqrt{x}}$

$$f(x) = \frac{1}{\sqrt[3]{x^2}} = x^{-\frac{2}{3}} \qquad f'(x) = -\frac{2}{3} \cdot x^{-\frac{5}{3}} = -\frac{2}{3} \cdot \frac{1}{\sqrt[3]{x^5}}$$

$$f(x) = 1 \cdot x \qquad f'(x) = 1 \cdot \overset{1}{x^0} = 1$$

$$f(x) = \underline{2}x \qquad f'(x) = \underline{2}$$

$$f(x) = 4\underline{x^2} \qquad f'(x) = 4 \cdot 2x = 8x$$

$$f(x) = 7x^3 \qquad f'(x) = 7 \cdot 3x^2 = 21x^2$$

$$f(x) = -4 \cdot x^7 \qquad f'(x) = -28 \cdot x^6$$

$$f(x) = \frac{3}{x} = 3 \cdot \frac{1}{x} = 3 \cdot x^{-1} \qquad f'(x) = 3 \cdot (-1) \cdot x^{-2} = -3 \cdot \frac{1}{x^2} = -\frac{3}{x^2}$$

$$f(x) = 4 \cdot \sqrt[6]{x^5} = 4 \cdot x^{\frac{5}{6}} \qquad f'(x) = \overset{2}{\cancel{4}} \cdot \frac{5}{\underset{3}{\cancel{6}}} \cdot x^{-\frac{1}{6}} = \frac{10}{3} \cdot \frac{1}{\sqrt[6]{x}}$$

$$f(x) = \frac{x^3}{3} = \frac{1}{3} \cdot x^3 \qquad f'(x) = \frac{1}{\cancel{3}} \cdot \cancel{3}x^2 = x^2$$

$$f(x) = x^3 + 2x^2 \qquad f'(x) = 3x^2 + 4x$$

$$f(x) = 4x^5 + 2x^2 - 7x - 3 \qquad f'(x) = 20x^4 + 4x - 7$$

$$f(x) = -5\sqrt{x} + \frac{8}{x^2} + 8 \qquad f'(x) = -5 \cdot \frac{1}{2} \cdot x^{-\frac{1}{2}} + 8 \cdot (-2) \cdot x^{-3}$$

$$-5 \cdot x^{\frac{1}{2}} + 8x^{-2} + 8 \qquad = -\frac{5}{2} \cdot \frac{1}{\sqrt{x}} - \frac{16}{x^3}$$

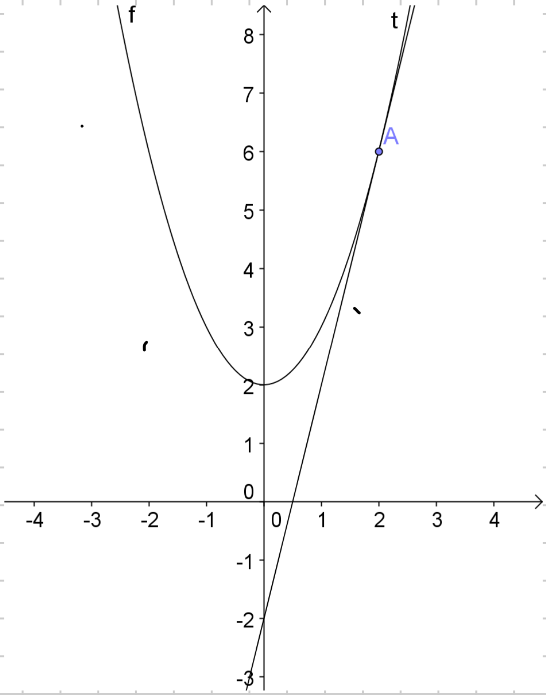
$$f(x) = 5^x \qquad f'(x) = 5^x \cdot \ln 5$$

$$f(x) = \ln(x) \qquad f'(x) = \frac{1}{x}$$

Bsp. 1.1. d $\quad f(x) = x^2 + 2 \quad x_0 = \underline{\underline{2}}$ $\quad \boxed{y}\boxed{=}\boxed{x^2+2}$

$f'(x) = 2x$

$k_t = f'(\underline{2}) = 2 \cdot 2 = \underline{\underline{4}}$



Zusatz: Gleichung der Tangente

$t: \quad y = k \cdot x + d$

$y_0 = f(x_0) = f(2) = 6$

$2^2 + 2$

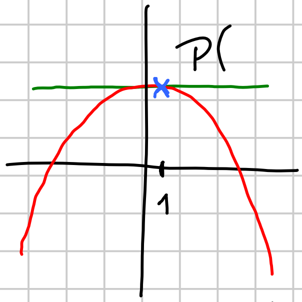
$6 = 4 \cdot 2 + d$

$6 = 8 + d$

$-2 = d$

$\Rightarrow \underline{\underline{t: \ y = 4 \cdot x - 2}}$

Bsp. 1.2. b $\quad \underline{\underline{f(x) = -x^2 + 2x - 4}}$



waagr. Tangente ?

$\Rightarrow$ Tangente mit $k = 0$

$\Rightarrow f'(x) \stackrel{!}{=} 0$

$f'(x) = -2x + 2 \stackrel{!}{=} 0$

$2 = 2x$

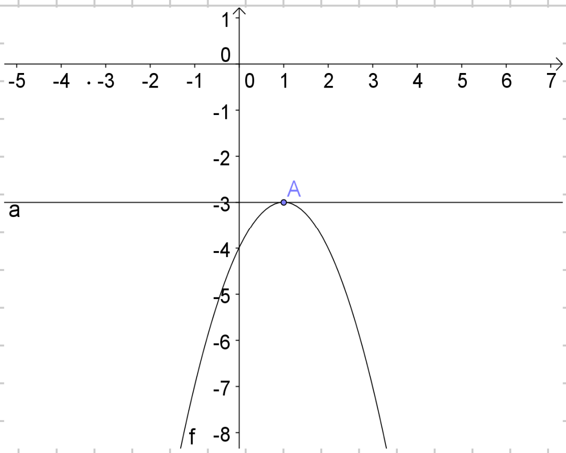
$\underline{\underline{1 = x}}$

$y = f(x)$

$y = f(1) = -1^2 + 2 \cdot 1 - 4 = \underline{\underline{-3}} \quad \Rightarrow P(1|-3)$

$y = \underset{0}{\underline{k}} x + d$

$t: \ y = -3$



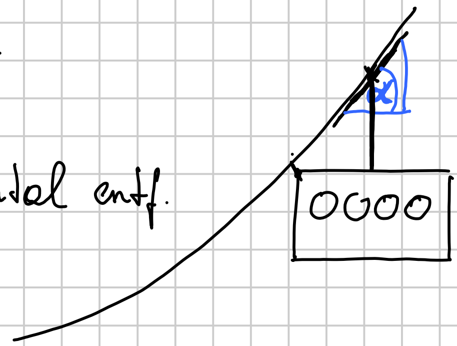
Bsp. 1.3.d) $f(x) = \frac{\sqrt{x}}{\sqrt[3]{x}} = \frac{x^{\frac{1}{2}}}{x^{\frac{1}{3}}} = x^{\frac{1}{2}-\frac{1}{3}} = x^{\frac{3}{6}-\frac{2}{6}} = x^{\frac{1}{6}} = \sqrt[6]{x}$

$f'(x) = \underline{\frac{1}{6} \cdot x^{-\frac{5}{6}}} = \frac{1}{6} \cdot \frac{1}{\sqrt[6]{x^5}}$

$f'(1{,}5) = \frac{1}{6} \cdot (1{,}5)^{-(5/6)} = 0{,}118\ldots \approx \underline{\underline{0{,}119}}$

Bsp. Seilbahn Seil verläuft gemäß der Fkt

$f(x) = 0{,}00028 \cdot x^2 + 1702$

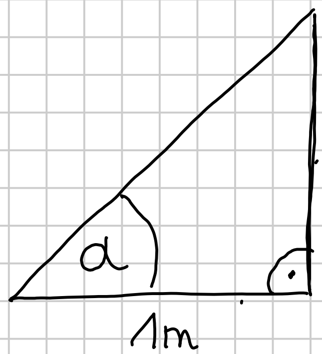


Bergstation: 1900m horizontal entf.
$x = 1900\,m$

? Winkel des Seiles

↳ Steigung $k = f'(x) = 0{,}00056\,x$

$f'(1900) = 0{,}00056 \cdot 1900 = 1{,}064$

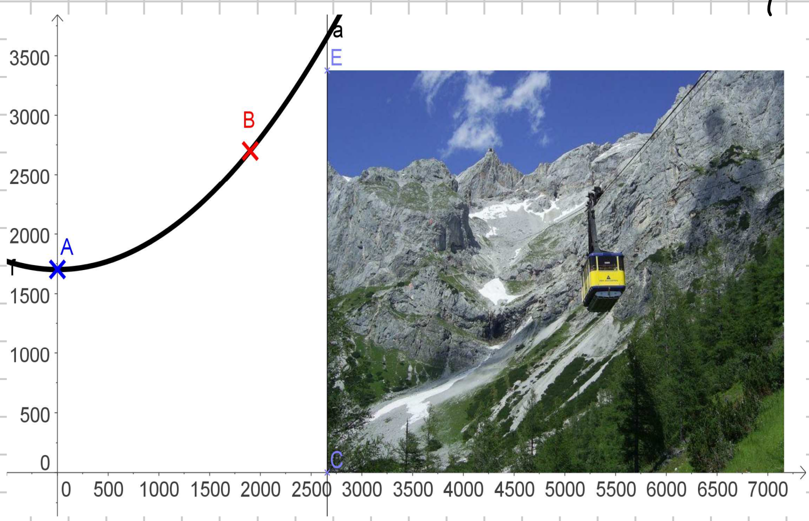


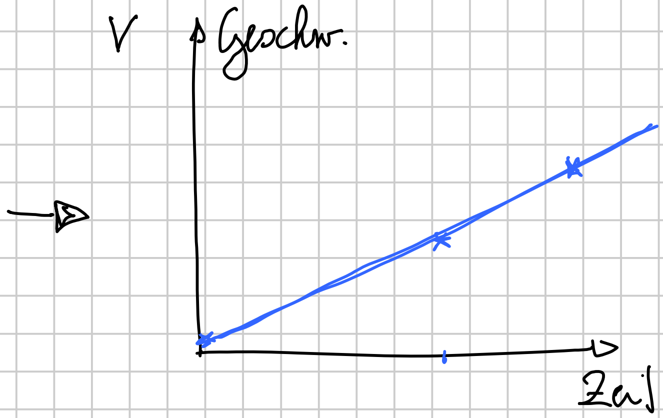
$\tan\alpha = \frac{GK}{AK} = \frac{k}{1}$

$\boxed{\tan\alpha = k}$

$\alpha = \arctan k$

$\alpha = \arctan 1{,}064 = 46{,}7°$





$\frac{\text{Weg}}{\text{Zeit}}$ - Geschw. $\frac{m}{s}$

$\frac{\text{Geschw.}}{\text{Zeit}}$ = Beschl. [a] $\frac{m}{s^2}$

$\Rightarrow \quad f(x) = x^3 \qquad f'(x) = 3x^2 \qquad f''(x) = 6x$

S. 9/15

Bsp. 1.8a) $f(x) = 0{,}5 \cdot x^3 \qquad P(-3, y_0)$

Tangente $\quad y = \underline{k}x + \underline{d} \qquad f(-3) = 0{,}5 \cdot (-3)^3 = \underline{\underline{-13{,}5}}$

$k_t = k_f$ = 1. Ableitung $\qquad f'(x) = 0{,}5 \cdot 3 \cdot x^2$

$\underline{\underline{f'(x) = 1{,}5x^2}}$

$k_t = f'(-3) = 1{,}5 \cdot (-3)^2 = 13{,}5$

$y_p = k \cdot x_p + d$

$-13{,}5 = 13{,}5 \cdot (-3) + d \quad / +40{,}5$

$\underline{\underline{27 = d}}$

$\Rightarrow \quad \underline{\underline{t_p: \; y = 13{,}5 \cdot x + 27}}$

Zusatz: Krümmung = Änderung der Steigung
= 2. Ableiten

? Wo ist die Krümmung 0 ?

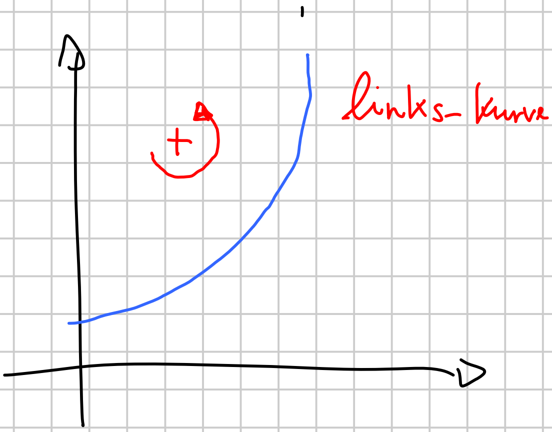
$f'(x) = 1{,}5x^2$, $f''(x) = 3x \overset{!}{=} 0$ $\qquad 0{,}5 \cdot 0^3$

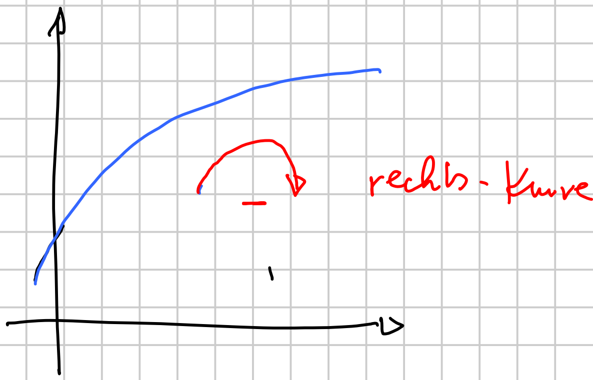
$x = 0 \quad \Rightarrow y = 0$

$P(0|0)$

Allg.: Krümmung

$+$ $\qquad\qquad$ $-$





S. 9/27 $\quad$ Bsp. $\quad f(x) = \frac{1}{6} \cdot (x^3 + x^2 - 16x - 16)$

Nullstellen: $\quad \frac{1}{6} \cdot (x^3 + x^2 - 16x - 16) \overset{!}{=} 0 \quad | \cdot 6$

$$x^3 + x^2 - 16x - 16 = 0$$

$\Rightarrow$ TI82: SOLVER $\qquad$ Startwert 0

$\Rightarrow x_1 = -1$

$$(x^3 + x^2 - 16x - 16) : (x + 1) = \underline{\underline{x^2 - 16}}$$
$$\underline{-x^3 \mp x^2}$$
$$0 \; -16x - 16$$
$$\underline{\pm 16x \pm 16}$$
$$0\,R$$

$\hookrightarrow \quad x^2 - 16 = 0$

$x^2 = 16$

$x = \pm 4$

$x_2 = +4$

$x_3 = -4$

2. Extrema: Min / Max

↳ $k = 0 = f'(x) = \frac{1}{6} \cdot (3x^2 + 2x - 16) \stackrel{!}{=} 0 \quad / \cdot 6$

$$3x^2 + 2x - 16 = 0$$

→ ABC - Formel

$A = 3$
$B = 2$
$C = -16$

$x_1 = -2{,}\dot{6}$
$x_2 = 2$