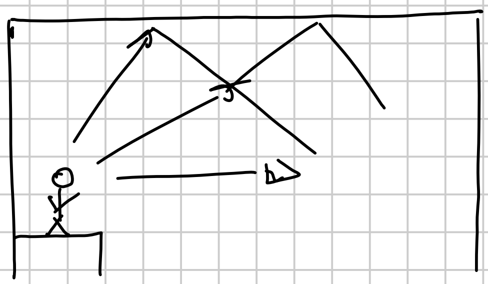
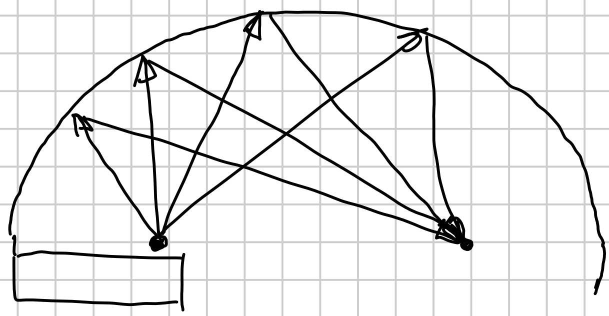
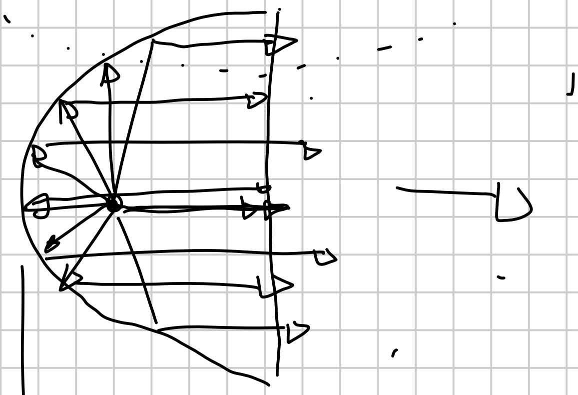
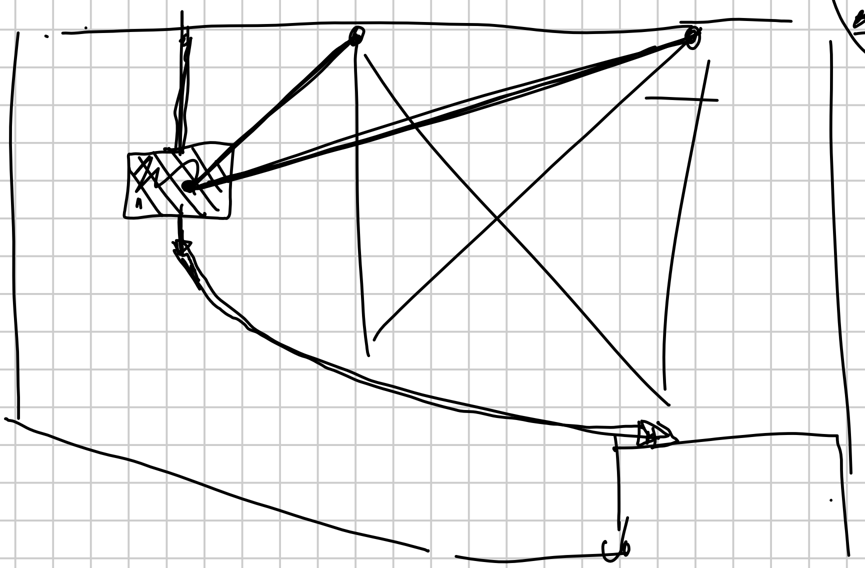
Differentialrechnung (3)

„Aufwärmen“ – Wozu Differentialrechnung









→ Bsp. NEWTON-Näherungsverfahren

$x^3 + 2x + 5 = 0 \quad \Rightarrow$ Lösung

http://home.eduhi.at/teacher/alindner/Dyn_Geometrie/DiffInt/NewtonNaeherung.htm

Kurvendiskussion

http://members.chello.at/gut.jutta.gerhard/kurs/kd.htm

Kurvendiskussion

S. 9/27

$$f(x) = \frac{1}{6} \cdot (x^3 + x^2 - 16x - 16)$$

1) Definitionsmenge

Polynomfkt.

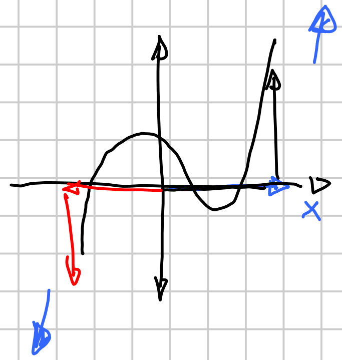
$$D = \mathbb{R}$$

2) Asymptotisches Verhalten

(Verhalten um $\pm\infty$)

$$\lim_{x \to \infty} f(x) = +\infty$$

$$\lim_{x \to -\infty} f(x) = -\infty$$



3) Ableitungen

$$f'(x) = \frac{1}{6} \cdot (3x^2 + 2x - 16)$$

$$f''(x) = \frac{1}{6} \cdot (6x + 2)$$

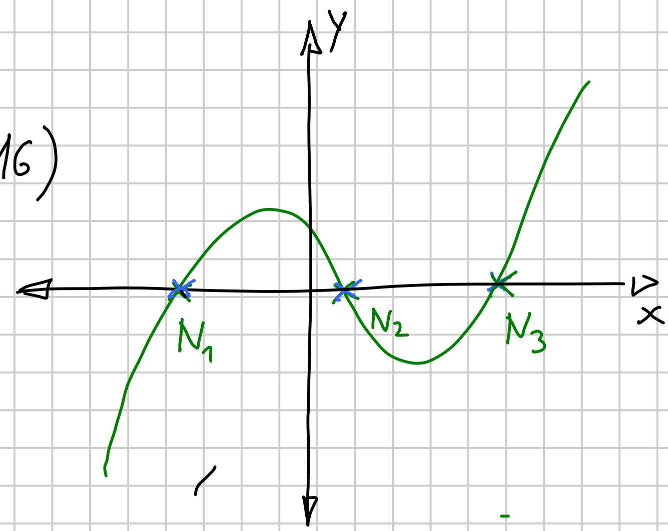
$$f'''(x) = \frac{1}{6} \cdot 6 = 1$$

4) Nullstellen

$f(x) = 0 = y$

$$f(x) = 0 \overset{!}{=} \cancel{\frac{1}{6}} \cdot (x^3 + x^2 - 16x - 16)$$

$$0 = x^3 + x^2 - 16x - 16$$



mögl. Lösungen $\pm 1, \pm 2, \pm 4, \pm 8, \pm 16$

? $\cancel{x = 1}$: $1^3 + 1^2 - 16 \cdot 1 - 16 \neq 0$

$x = -1$: ✓ $(-1)^3 + (-1)^2 - 16 \cdot (-1) - 16$

$-1 + 1 + 16 - 16 = 0$ ✓

sonst TI82: MATH – SOLVER: $0 = x^3 + x^2 - 16x - 16$

Startwert $x = 0 \Rightarrow \underline{\underline{x_1 = -1}}$

Nullstelle

$\underline{\underline{N_1 = (-1 | 0)}}$

$\Rightarrow \forall$ 2 Nullstellen

$$x - (-1)$$

$$(x^3 + x^2 - 16x - 16) : (x + 1) = \underline{x^2 - 16}$$

$$\underline{-x^3 \mp x^2}$$

$$/ \quad / \quad -16x - 16$$

$$\underline{\pm 16x \pm 16}$$

$$0R$$

$\Rightarrow x^2 - 16 = 0$

$$x^2 = 16$$

$$x_2 = -4$$

$$x_3 = +4$$

$\underline{\underline{N_2 (-4 | 0)}}$ $\underline{\underline{N_3 (4 | 0)}}$

5) Extrempunkte (Hochpunkt / Tiefpunkt)

Steigung = 0

= 1. Ableitung

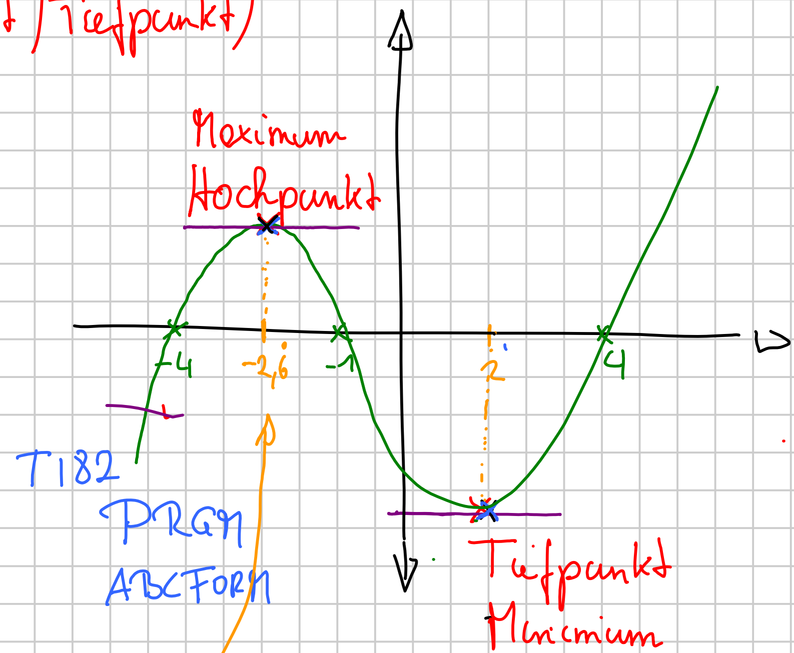
$$\cancel{\tfrac{1}{6}} \cdot (3x^2 + 2x - 16) \overset{!}{=} 0 \quad | : \tfrac{1}{6}$$

$\rightarrow$ ABC-Formel

$A = 3$

$B = 2$

$C = -16$



$$x_1 = 2$$

$$x_2 = -2,\dot{6} = -\frac{8}{3}$$

$$y_1 = f(2) = \frac{1}{6} \cdot (2^3 + 2^2 - 16 \cdot 2 - 16)$$

$$= -6$$

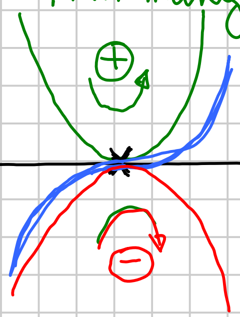
$$y_2 = f(-2,\dot{6}) = \ldots = 2,47$$

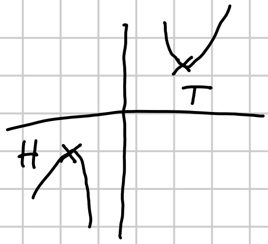
$E_1 (2 | -6)$

$E_2 (-2,\dot{6} | 2,47)$

Nachweis über H bzw. T / Mini - Max

⇒ Krümmung





$f''(x) > 0$ ⇒ Tiefpunkt / Minimum

$f''(x) = 0$ W+E=S Steigung

$f''(x) < 0$ ⇒ Hochpunkt / Maximum

? pos. od neg. Krümmung = 2. Ableitung !

$$f''(x) = \frac{1}{6} \cdot (6x+2)$$

$E_1(-2,\dot{6} \mid 2,47)$: $\quad -2,\dot{6} = -\frac{8}{3}$

$$f''(-2,\dot{6}) = f''\left(-\frac{8}{3}\right) = \frac{1}{6} \cdot \left(\cancel{6}^{\,2} \cdot \left(-\frac{8}{\cancel{3}}\right) + 2\right) = -\frac{14}{6} = -\frac{7}{3}$$

< 0 negativ

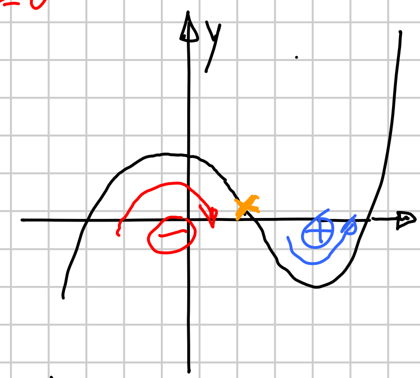
⇒ $\underline{\underline{\text{Hochpunkt } H(-2,\dot{6} \mid 2,47)}}$

$E_2(2 \mid -6)$:

$$f''(2) = \frac{1}{6} \cdot (6 \cdot 2 + 2) > 0$$

⇒ Tiefpunkt $T(2 \mid -6)$

(6) Wendepunkte / Wendestellen $f''(x) = 0$



2. Ableitung = Krümmung = 0

$$f''(x) = \cancel{\frac{1}{6}} \cdot (6x+2) \overset{!}{=} 0 \quad | -2$$

$$6x = -2 \quad | :6$$

$$x_w = -\frac{1}{3} \quad \text{(Wendestelle)}$$

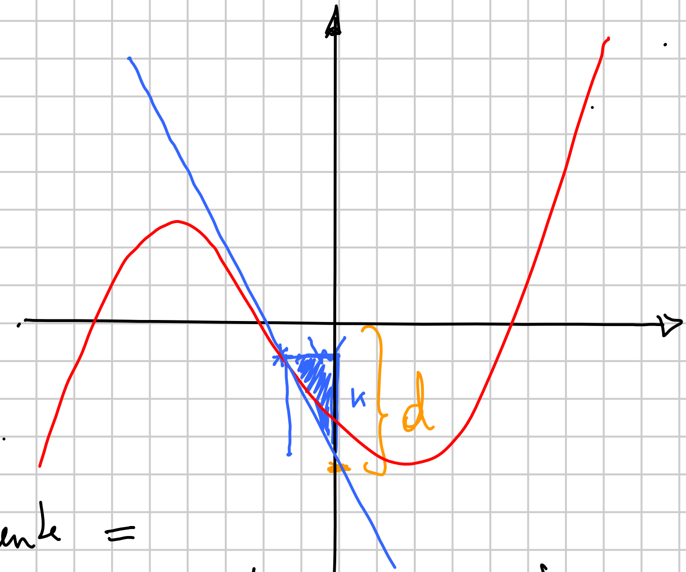
$$y = f\left(-\frac{1}{3}\right) = \left[\frac{1}{6} \cdot \left(\left(-\frac{1}{3}\right)^3 + \left(-\frac{1}{3}\right)^2 - 16 \cdot \left(-\frac{1}{3}\right) - 16\right)\right] = \quad \text{TI 82}$$

$$= \underline{\underline{-1,77}} \quad ⇒ \quad \underline{\underline{W\left(-\frac{1}{3} \mid -1,77\right)}}$$

Wendepunkt

(7) Wendetangente

$t_w: \quad y = kx + d$



$W(-\frac{1}{3} | -1{,}77) \in t_w$

x_w

$-1{,}77 = k \cdot (-\frac{1}{3}) + d$

$k_w = f'(x_w)$ Steigung Tangente = Steigung der Fkt am Wendepunkt

$k_w = f'(-\frac{1}{3}) = \left[\frac{1}{6} \cdot (3 \cdot (-\frac{1}{3})^2 + 2 \cdot (-\frac{1}{3}) - 16)\right.$

$\left. = \frac{1}{6} \cdot (\frac{1}{3} - \frac{2}{3} - 16)\right] = \underline{-2{,}72}$

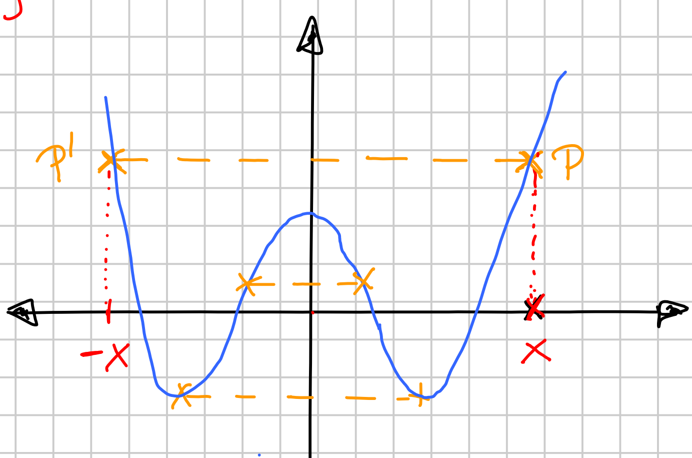
!

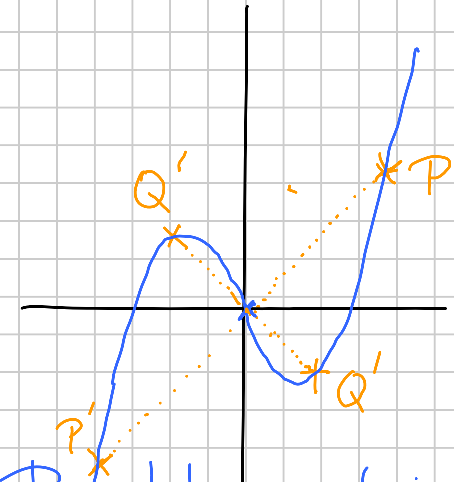
$\Rightarrow \quad -1{,}77 = -2{,}72 \cdot (-\frac{1}{3}) + d$

$\underline{-2{,}67 = d}$

$\Rightarrow \underline{\underline{t_w: \quad y = -2{,}72 \cdot x - 2{,}67}}$

(8) Symmetrische Verhalten





Axialsymmetrisch bzgl. y-Achse

$f(x) = f(-x)$

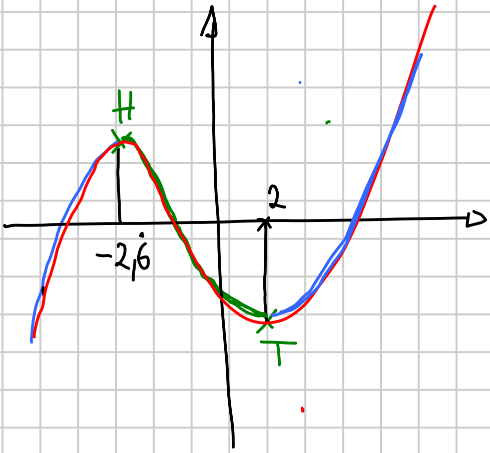
$\Rightarrow$ gerade Fkt (x^{2n}) $2n$ gerade $2, 4, 6, 8, 10, \ldots$

Punktsymmetrisch bzgl. Ursprung

$f(-x) = -f(x)$

$\Rightarrow$ ungerade Fkt (x^{2n-1}) $1, 3, 5, 7, \ldots$

(9) Monotonieverhalten

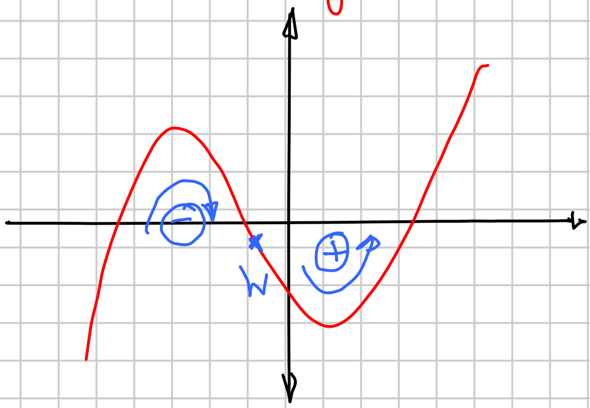


$]-\infty; -2,\dot{6}]$ monoton wachsend

$]-2,\dot{6}; 2]$ monoton fallend

$]2; \infty[$ monoton wachsend

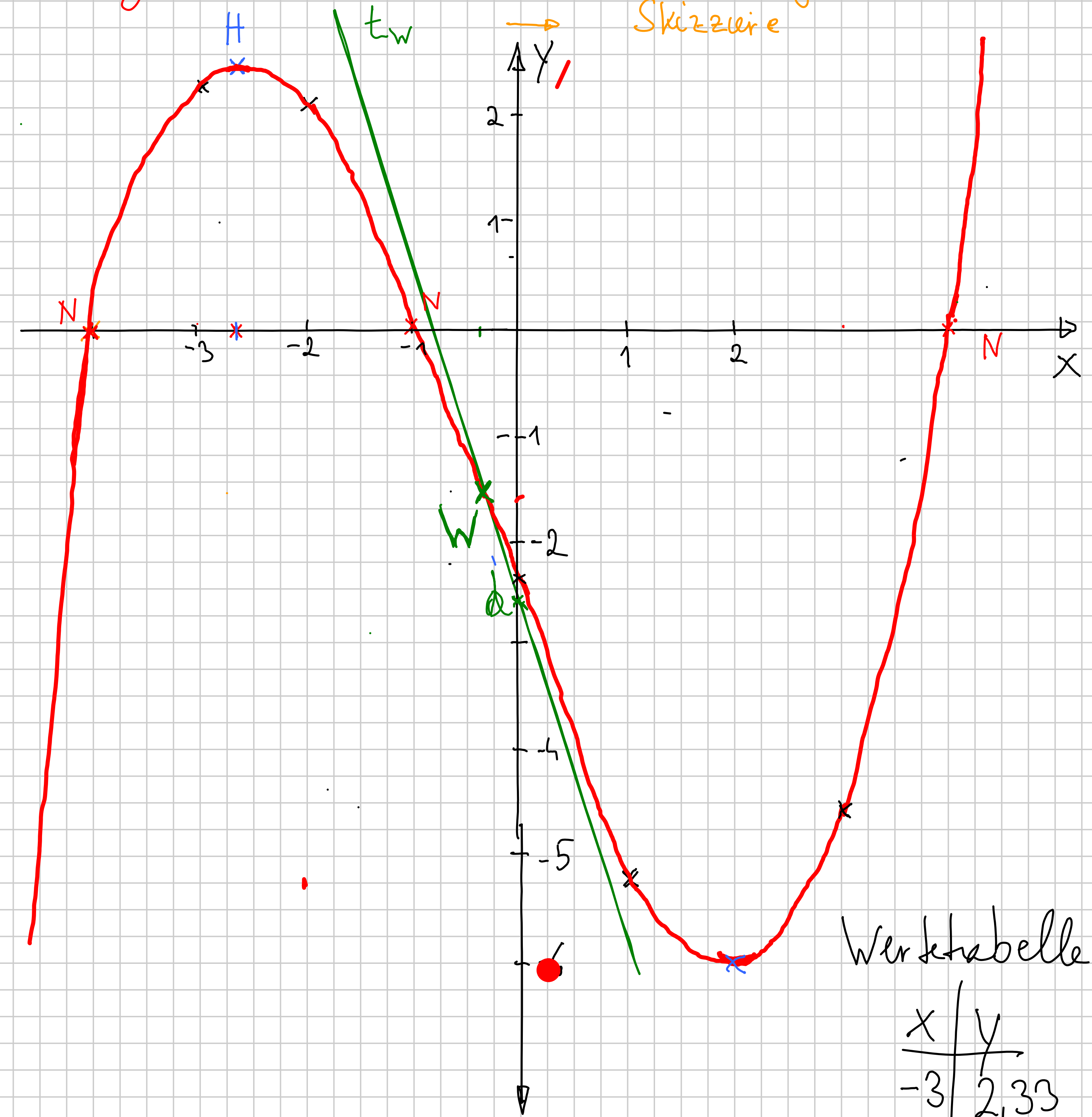
(10) Krümmungsverhalten



$]-\infty; -\frac{1}{3}]$ neg. Krümm.

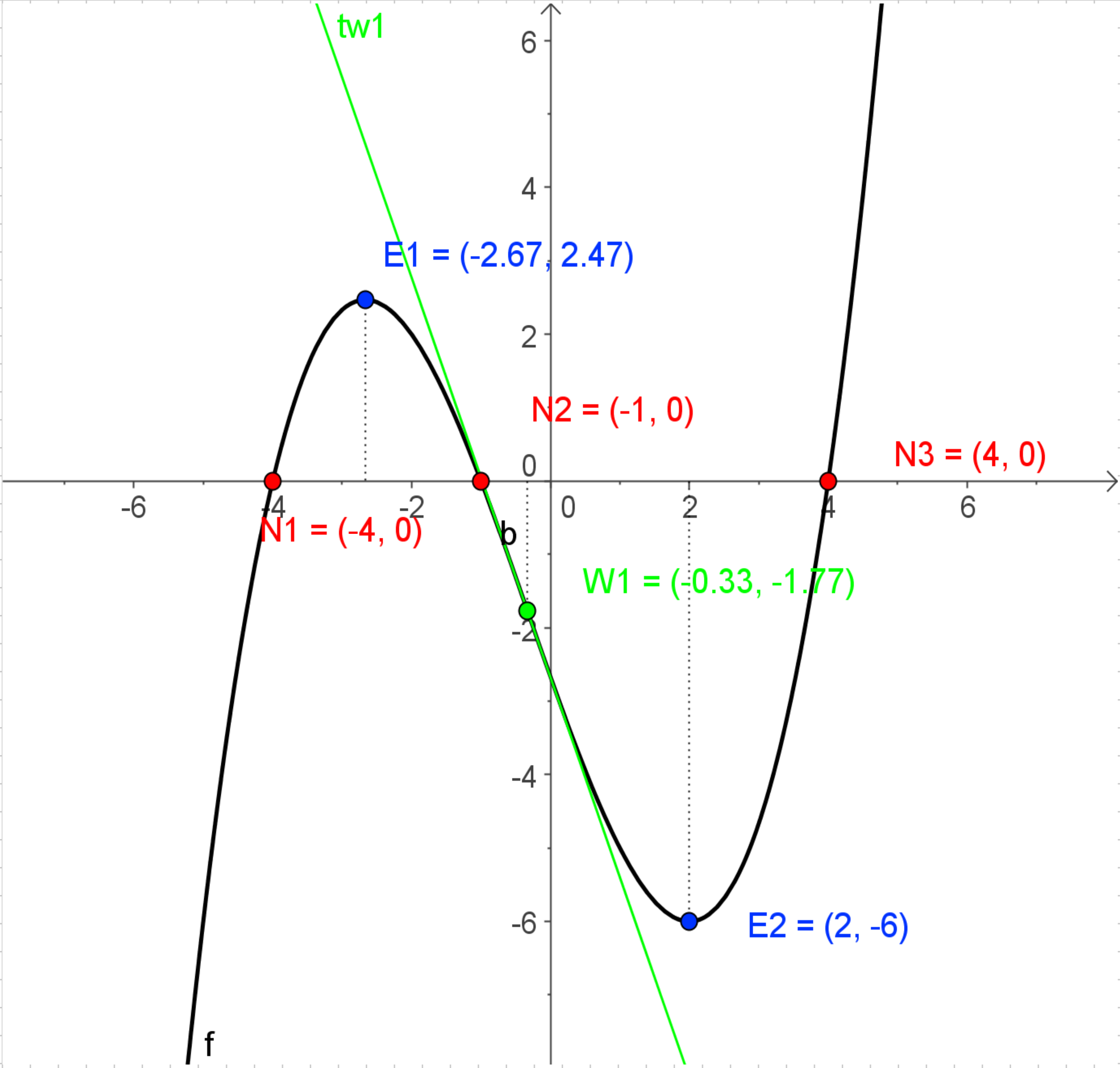
$]-\frac{1}{3}; \infty[$ pos. Krümm.

(11) Graph der Fkt → Zeichnung (exakt)
→ Skizziere



Wertetabelle

x	y
−3	2,33
−2	2
−1	0
0	−2,6
1	−5
2	−6
3	−4,6


tw1
E1 = (-2.67, 2.47)
N2 = (-1, 0)
N3 = (4, 0)
N1 = (-4, 0)
b
W1 = (-0.33, -1.77)
E2 = (2, -6)
f
6
4
2
0
-2
-4
-6
-6
-4
-2
0
2
4
6

Diskutiere die Fkt $f(x) = \frac{x^2-4}{3x-10}$

gebr. rationale Fkt.

(1) Definitionsmenge

$D = \mathbb{R} \setminus \{\frac{10}{3}\}$

ohne

$3x - 10 \neq 0$

$3x \neq 10$

$x \neq \frac{10}{3}$

(2) Asymptotische Verhalten

$\lim\limits_{x \to \infty} f(x) = \lim\limits_{x \to \infty} \frac{x^2-4}{3x-10} = \infty$

$\lim\limits_{x \to -\infty} f(x) = \lim\limits_{x \to -\infty} \frac{x^2-4}{3x-10} = -\infty$

$+\infty$, $-\infty$

Zählergrad 2

Nennergrad 1

ZG > NG $\to \infty$

$\to$ Polynomdiv. $\frac{x^2-4}{3x-10}$ = $x^2 \quad -4 : 3x - 10 = \boxed{\frac{1}{3}x + \frac{10}{9}}$

$-\; x^2 \pm \frac{10}{3}x$

$+\frac{10}{3}x - \cancel{4}\ \frac{36}{9}$

$\mp \frac{10}{3}x \pm \frac{100}{9}$

$\frac{64}{9}$ R

$3x \cdot \square = \frac{10}{3}x \quad |:3$

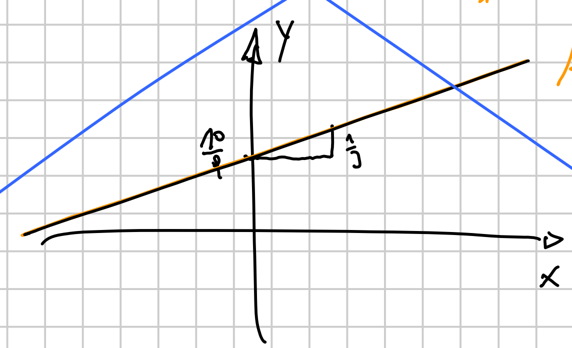
$\square = \frac{10}{9}$

$25 : 7 = 3$
4R

$\frac{25}{7} = 3 + \frac{4}{7}$

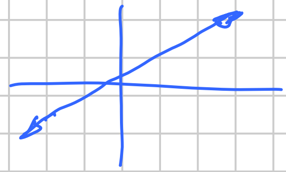
$\frac{x^2-4}{3x-10} = \boxed{\frac{1}{3}x + \frac{10}{9}} + \frac{\frac{64}{9}}{3x-10}$

$\lim\limits_{x \to \infty} \frac{1}{3}x + \frac{10}{9} + \frac{\frac{64}{9}}{3x-10}$ → 0



Asymptote

⇒ schräge Asymptote $\boxed{y = \frac{1}{3}x + \frac{10}{9}}$



⇒ $\lim_{x \to \infty} f(x) = +\infty$

$\lim_{x \to -\infty} f(x) = -\infty$

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vgl S. 9/33

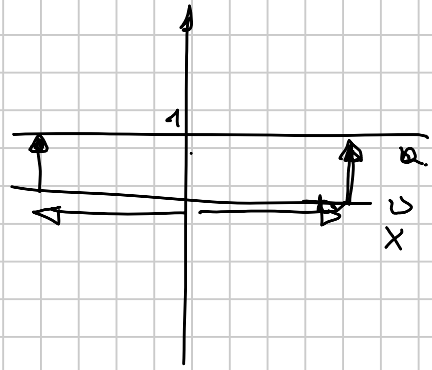
$f(x) = \frac{x^2+2}{x^2-4}$

$x^2+2 : x^2-4 = 1$

$-x^2 + 4$

$+6\ R$

$\underline{\underline{y = 1}}$



$$\boxed{\begin{array}{l} \lim_{x \to \infty} f(x) = 1 \\ \lim_{x \to -\infty} f(x) = 1 \end{array}}$$

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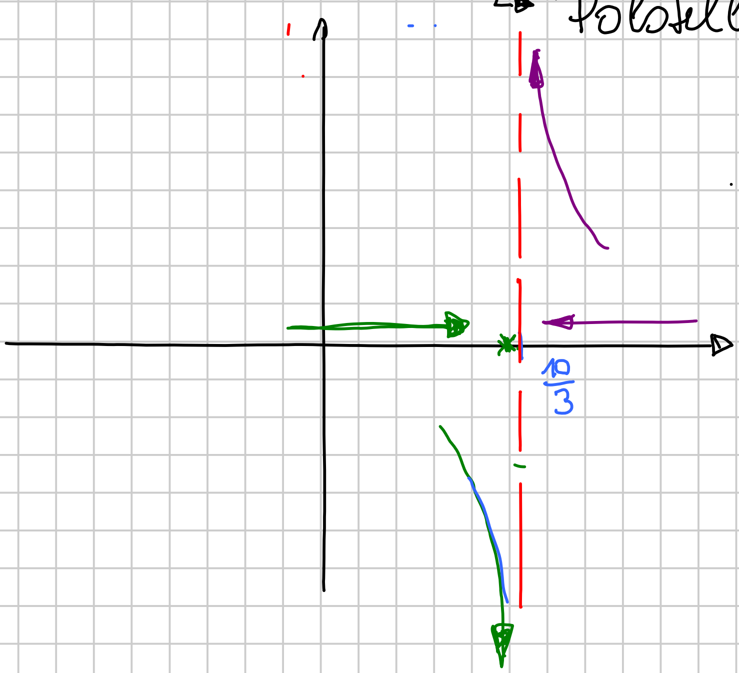
aus (1)' $D = \mathbb{R} \setminus \{\frac{10}{3}\}$

↳ Polstelle ⇒ senkr. Asymptote

a: $x = \frac{10}{3}$



$\lim_{x \to \frac{10}{3}^{(-)}} \frac{\overset{pos}{x^2-4}}{\underset{neg}{3x-10}} = -\infty$

$\lim_{x \to \frac{10}{3}^{(+)}} \frac{\overset{pos}{x^2-4}}{\underset{pos}{3x-10}} = +\infty$

$\frac{f' \cdot g - f \cdot g'}{g^2}$

(3) Ableitungen

Quotient

$$f(x) = \frac{x^2-4}{3x-10}$$

$$f = x^2-4 \qquad f' = 2x$$

$$g = 3x-10 \qquad g' = 3$$

$$f'(x) = \frac{2x\cdot(3x-10) - (x^2-4)\cdot 3}{(3x-10)^2} = \frac{6x^2-20x-3x^2+12}{(3x-10)^2} =$$

$$f'(x) = \frac{3x^2-20x+12}{(3x-10)^2}$$

$$f = 3x^2-20x+12 \qquad f' = 6x-20$$

$$g = (3x-10)^2 \qquad g' = 2\cdot\underbrace{(3x-10)^1\cdot 3}_{6}$$

$$f''(x) = \frac{(6x-20)\cdot(3x-10)^{\cancel{2}} - (3x^2-20x+12)\cdot 6\cdot\cancel{(3x-10)^1}}{(3x-10)^{\cancel{4}\,3}}$$

$$f''(x) = \frac{\cancel{18x^2}\cancel{-60x}\cancel{-60x}+200-\cancel{18x^2}\cancel{+120x}-72}{(3x-10)^3}$$

$$f''(x) = \frac{128}{(3x-10)^3}$$

(4) Nullstellen

$f(x)=0=y$

$$f(x) = \frac{x^2-4}{3x-10} = 0 \quad |\cdot(3x-10)$$

$$x^2-4 = 0$$

$$x^2 = 4$$

$$x_1 = -2 \qquad x_2 = +2$$

$$N_1(-2|0) \qquad N_2(2|0)$$

(5) Extrempunkte

$f'(x)=0$

$$f(x) = \frac{3x^2-20x+12}{(3x+10)^2} \overset{!}{=} 0 \quad |\cdot N$$

$$3x^2 - 20x + 12 = 0$$

$\rightarrow$ TI 82   ABC-Formel $A = 3$, $B = -20$, $C = 12$

$x_1 = 6$ $\boxed{X}$

$x_2 = 0{,}\dot{6} = \frac{2}{3}$ $\boxed{Y}$

$$f(6) = \frac{6^2 - 4}{3 \cdot 6 - 10} = \frac{32}{8} = 4 \qquad E(6|4) \qquad f''(6) > 0 \Rightarrow T$$

$$f\left(\tfrac{2}{3}\right) = \quad = \frac{4}{9} = 0{,}\dot{4} \qquad E\left(\tfrac{2}{3}\middle|\tfrac{4}{9}\right) \qquad f''\left(\tfrac{2}{3}\right) < 0 \Rightarrow H$$

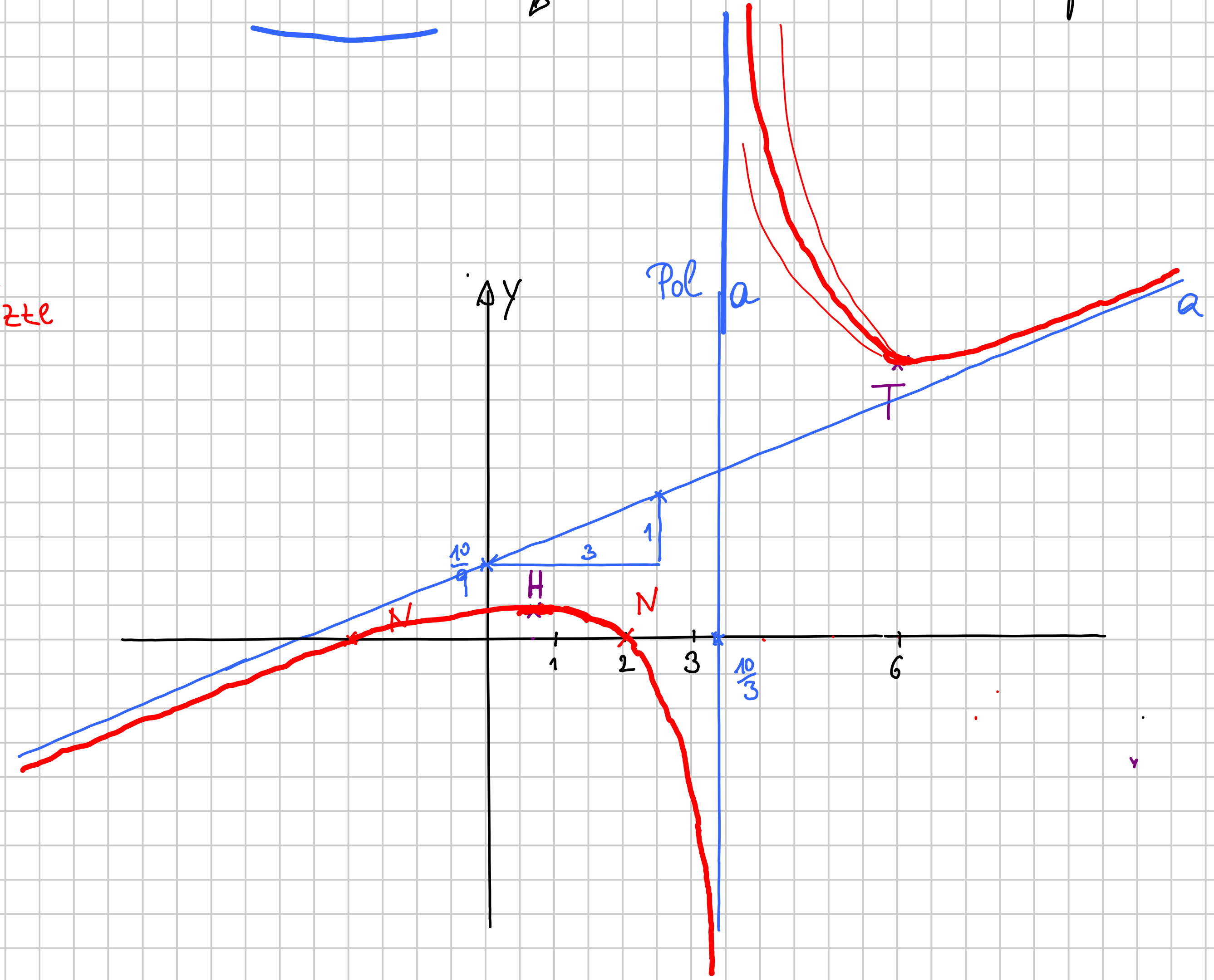
(6) Wendepunkt $f''(x) = 0$

$$f''(x) = \frac{128}{(3x-10)^3} = 0$$

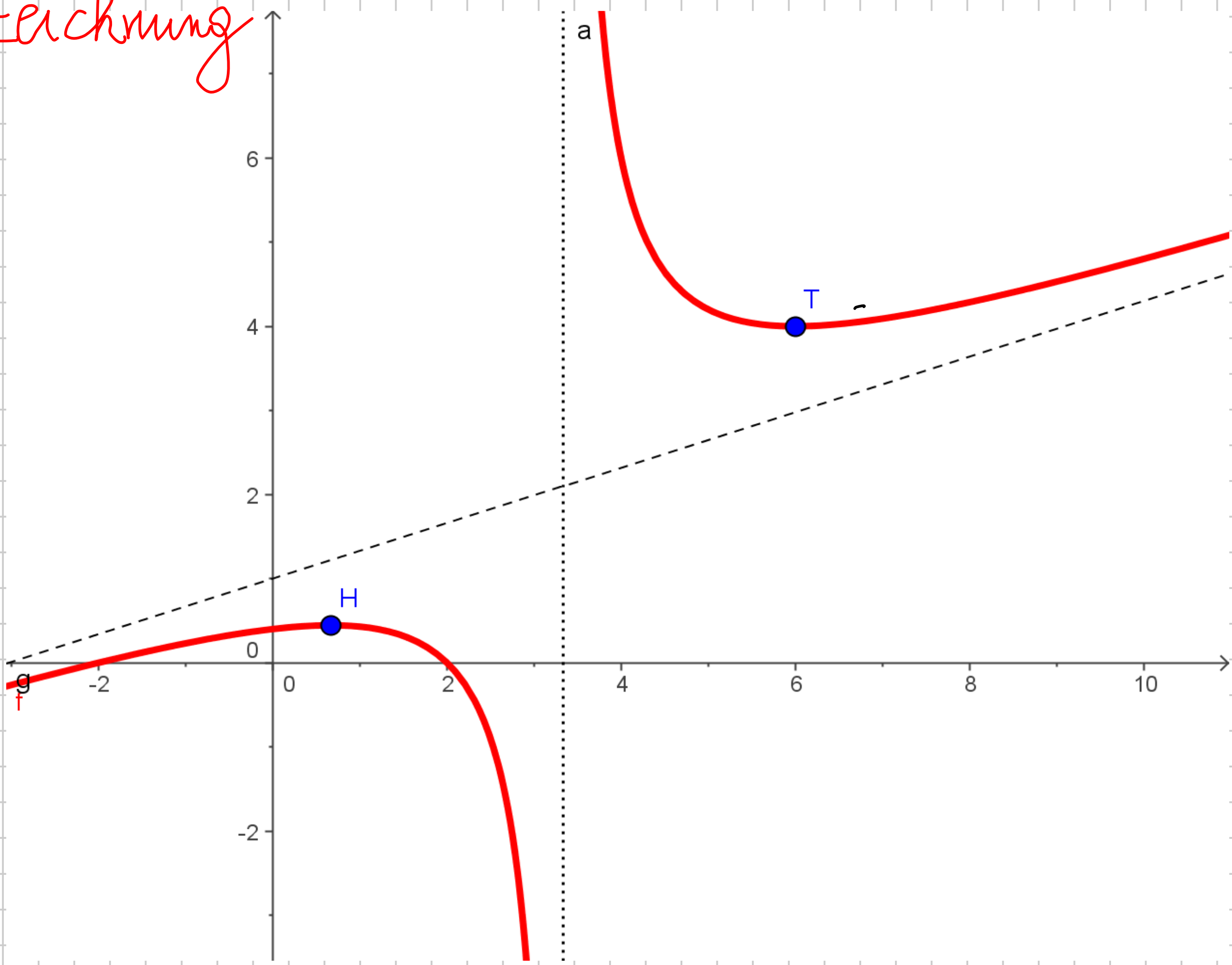
$$\underline{128 = 0} \quad ↯ \quad \Rightarrow \text{kein Wendepunkt}$$

(7) X

(8) Skizze



Zeichnung



Diskutiere die Funktion $f(x) = \dfrac{x^3}{(x-2)^2} = \dfrac{x^3}{x^2-4x+4}$

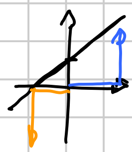
(1) Definitionsmenge

$$D = \mathbb{R} \setminus \{2\}$$

(2) Asy. Verh.

ZG > NG

$$x^3 : (x^2-4x+4) = x + 4$$
$$\underline{-x^3 \pm 4x^2 \mp 4x}$$
$$/ \quad +4x^2 - 4x$$
$$\underline{\mp 4x^2 \pm 16x \mp 16}$$
$$/ \quad +12x - 16 \; R$$

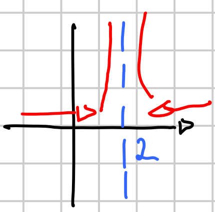


$\Rightarrow$ asy: $y = x + 4$

$$\lim_{x \to \infty} f(x) = +\infty$$

$$\lim_{x \to -\infty} f(x) = -\infty$$

$\Rightarrow$ Polstelle a: $x = 2$



$$\lim_{x \to 2^{(-)}} f(x) = +\infty$$

$$\lim_{x \to 2^{(+)}} f(x) = +\infty$$

(3) Ableitungen

$$f(x) = \frac{x^3}{(x-2)^2}$$

$f = x^3 \qquad f' = 3x^2$

$g = (x-2)^2 \qquad g' = 2(x-2)^1 \cdot 1$

$$f'(x) = \frac{3x^2(x-2)^{\cancel{2}} - x^3 \cdot 2\cancel{(x-2)}}{(x-2)^{\cancel{4}\,3}} = \frac{3x^3 - 6x^2 - 2x^3}{(x-2)^3}$$

$$f'(x) = \frac{x^3 - 6x^2}{(x-2)^3}$$

$f = x^3 - 6x^2 \qquad f' = 3x^2 - 12x$

$g = (x-2)^3 \qquad g' = 3(x-2)^2 \cdot 1$

$$f''(x) = \frac{(3x^2-12x)\cdot(x-2)^{\cancel{3}\,1} - (x^3-6x^2)\cdot 3\cancel{(x-2)}^{\cancel{2}}}{(x-2)^{\cancel{6}\,4}} =$$

$$f''(x) = \frac{\cancel{3x^3} - \cancel{12x^2} - \cancel{6x^2} + 24x - \cancel{3x^3} + \cancel{18x^2}}{(x-2)^4}$$

$$f''(x) = \frac{24x}{(x-2)^4}$$

(4) Nullstellen

$$\frac{x^3}{(x-2)^2} = 0 \quad \Rightarrow \quad x^3 = 0$$

$$x = 0 \qquad N(0|0)$$

(5) Extrempunkte

$$\frac{x^3 - 6x^2}{(x-2)^2} = 0 \quad \Rightarrow \quad x^3 - 6x^2 = 0$$

$$x^2 \cdot (x-6) = 0$$

$x^2 = 0$ oder $x - 6 = 0$

$x = 0$

$E(0|0) = N$

$f''(0) = 0$

Sattelpunkt

$x = 6$

$E(6|13{,}5)$

$f(6) = \frac{216}{16} = 13{,}5$

$f''(6) > 0$ Tiefpunkt

(6) Wendepunkt

$W(0|0) = S = E$