Differenzialrechnung (4)

Wiederholung

$$f(x) = -\frac{3}{4}x^4 - x^3 + 3x^2$$

1) Def - Menge $\quad D = \mathbb{R}$

2) Asympt. Verhalten

$$\lim_{x \to \infty} f(x) = -\infty$$

$$\lim_{x \to -\infty} f(x) = -\infty$$

3) Ableitungen

$$f'(x) = -\frac{3}{\cancel{4}} \cdot \cancel{4}x^3 - 3x^2 + 6x$$

$$f''(x) = -9x^2 - 6x + 6$$

4) Nullstellen $\quad f(x) = 0 = y$

$$-\frac{3}{4}x^4 - x^3 + 3x^2 = 0$$

$$x^2 \cdot \left(-\frac{3}{4}x^2 - x + 3\right) = 0$$

$$x^2 = 0 \qquad\qquad -\frac{3}{4}x^2 - x + 3 = 0$$

$$_2x_1 = 0$$

$\to$ T182 - ABC - Formel

$A = -\frac{3}{4} \qquad B = -1 \qquad C = 3$

$$x_3 = -2{,}77$$

$$x_4 = 1{,}44$$

$\underline{_2N_1(0|0)}$

$\underline{N_3(-2{,}77|0)}$

$\underline{N_4(1{,}44|0)}$

5) Extrempunkte

$$f'(x)=0 = -3x^3-3x^2+6x=0$$

$$3x(-x^2-x+2)=0$$

$3x=0$ $\qquad$ $-x^2-x+2=0$

$x=0$ $\qquad$ $\rightarrow$ ABC-Formel $\quad A=-1$

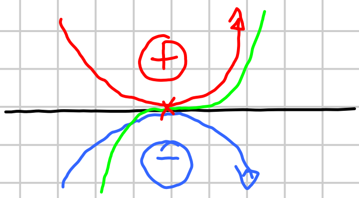
$B=-1$

$C=2$

$E_1(0|0)$ T

$E_2(-2|8)$ H $\qquad x_2=-2 \qquad y_2=f(-2)=8$

$E_3(1|\frac{5}{4})$ H $\qquad x_3=1 \qquad y_3=f(1)=\frac{5}{4}$



$f''(0)=6>0 \Rightarrow T$

$f''(-2)<0 \Rightarrow H$

$f''(1)<0 \Rightarrow H$

6) Wendepunkt

$$f''(x)=0=-9x^2-6x+6$$

TI82 $-$ ABC-Formel $A=-9 \quad B=-6 \quad C=6$

$x_1=-1{,}215\ldots \qquad y_1=f(-1{,}215)=4{,}59$

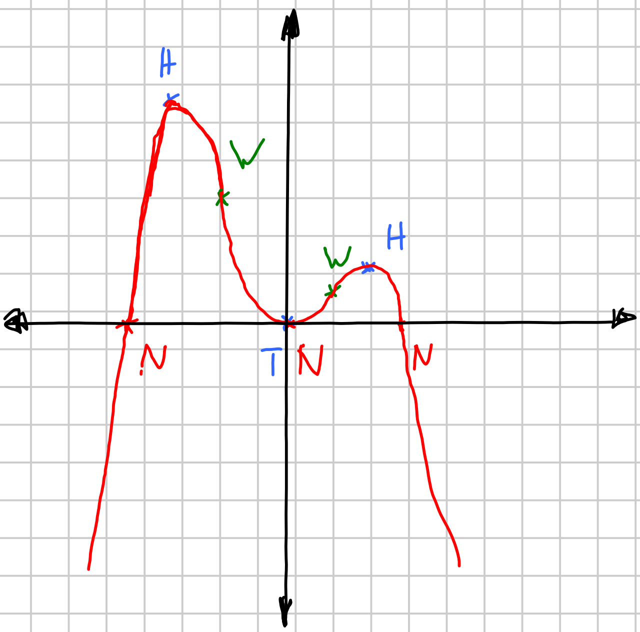
$\boxed{X}$

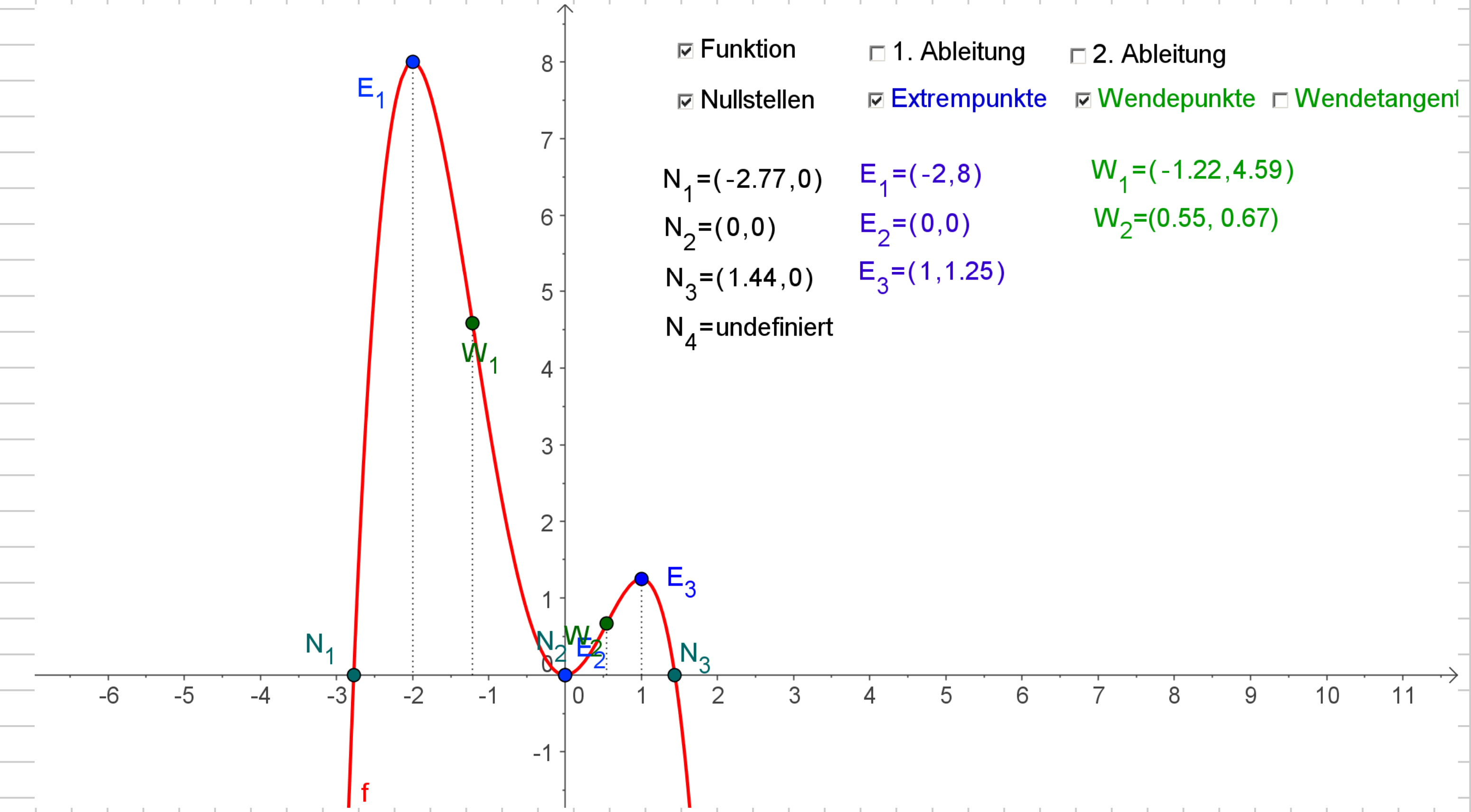
$x_2=0{,}548\ldots \qquad y_2=f(0{,}548)=0{,}67$

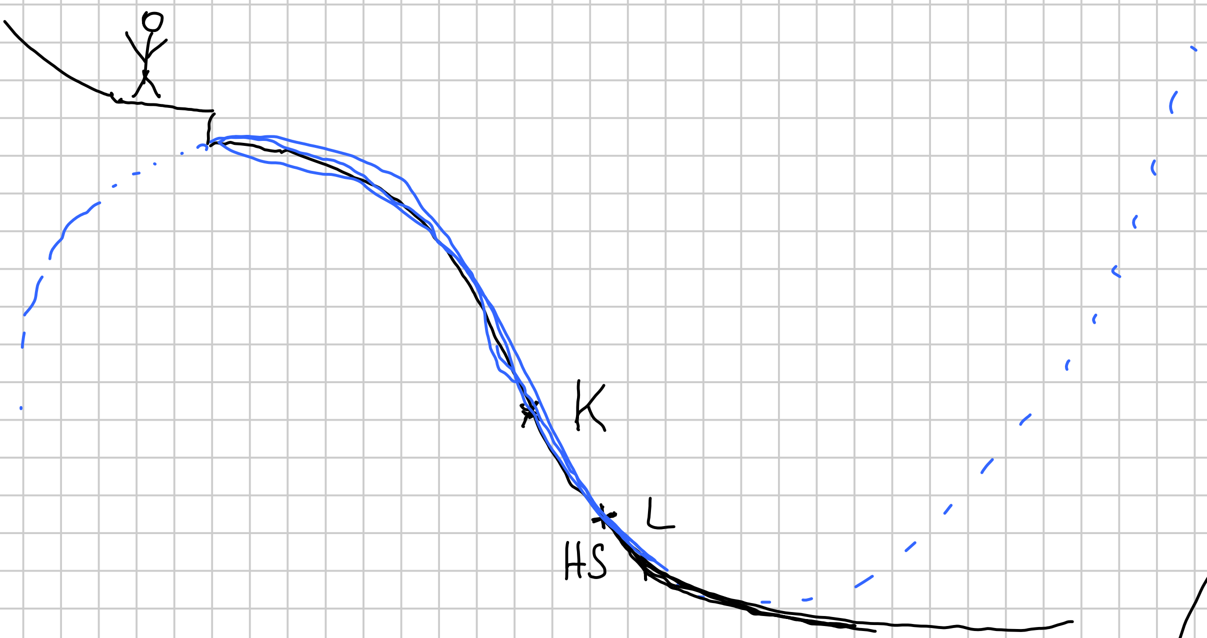
$\boxed{Y}$

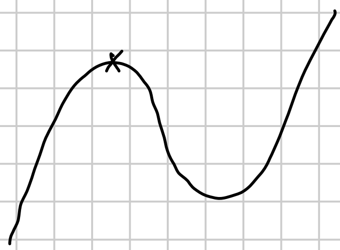
$W_1(-1{,}215|4{,}59)$

$W_2(0{,}548|0{,}67)$









Umkehraufgaben

WANTED: Fkt. 3. Grades $\rightarrow f(x) = a \cdot x^3 + b \cdot x^2 + c \cdot x + d$

Bsp. 2.4. $\boxed{H(1|7)}$, $T(3|1)$ $\quad f'(x) = 3ax^2 + 2bx + c$

I: $f(1) = 7 = a \cdot 1^3 + b \cdot 1^2 + c \cdot 1 + d$

II: $f(3) = 1 = a \cdot 3^3 + b \cdot 3^2 + c \cdot 3 + d$

III: $f'(1) = 0 = 3a \cdot 1^2 + 2b \cdot 1 + c$

IV: $f'(3) = 0 = 3a \cdot 3^2 + 2b \cdot 3 + c$

$$
\begin{aligned}
\text{I}: &\quad 1a + b + c + d = 7 \\
\text{II}: &\quad 27a + 9b + 3c + d = 1 \\
\text{III}: &\quad 3a + 2b + c = 0 \\
\text{IV}: &\quad 27a + 6b + c = 0
\end{aligned}
$$

$$
A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 27 & 9 & 3 & 1 \\ 3 & 2 & 1 & 0 \\ 27 & 6 & 1 & 0 \end{pmatrix} \qquad b = \begin{pmatrix} 7 \\ 1 \\ 0 \\ 0 \end{pmatrix}
$$

$$
A^{-1} \cdot b = \begin{pmatrix} 1{,}5 \\ -9 \\ 13{,}5 \\ 1 \end{pmatrix} \begin{array}{c} a \\ b \\ c \\ d \end{array}
$$

$$
\Rightarrow \underline{\underline{f(x) = 1{,}5x^3 - 9x^2 + 13{,}5x + 1}}
$$

B.2.6. $f(x) = ax^3 + bx^2 + cx + d$

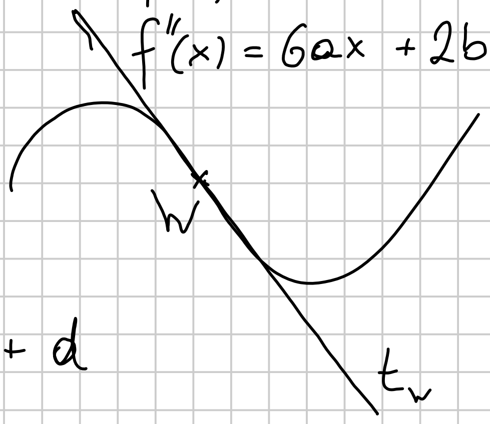
$f'(x) = 3ax^2 + 2bx + c$

$f''(x) = 6ax + 2b$

$H(\underline{2}|-2) \begin{cases} \to P \\ \to E \end{cases}$

$W(4|y_w) \qquad x_w = 4$

$t_w: \qquad k_w = -3$



$$
\begin{aligned}
\text{I}: &\quad f(2) = -2 = a \cdot 2^3 + b \cdot 2^2 + c \cdot 2 + d \\
\text{II}: &\quad f'(2) = 0 = 3a \cdot 2^2 + 2b \cdot 2 + c \\
\text{III}: &\quad f''(4) = 0 = 6a \cdot 4 + 2b \\
\text{IV}: &\quad f'(4) = -3 = 3a \cdot 4^2 + 2b \cdot 4 + c
\end{aligned}
$$

$$
A = \begin{pmatrix} 8 & 4 & 2 & 1 \\ 12 & 4 & 1 & 0 \\ 24 & 2 & 0 & 0 \\ 48 & 8 & 1 & 0 \end{pmatrix} \qquad b = \begin{pmatrix} -2 \\ 0 \\ 0 \\ -3 \end{pmatrix}
$$

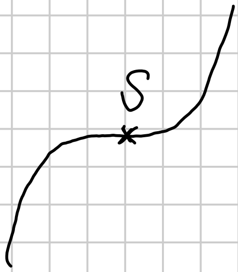
T182

$$
A^{-1} \times b = \begin{pmatrix} 0{,}25 \\ -3 \\ +9 \\ -10 \end{pmatrix}
$$

$$
f(x) = 0{,}25x^3 - 3x^2 + 9x - 10
$$

Bsp. 2.8.

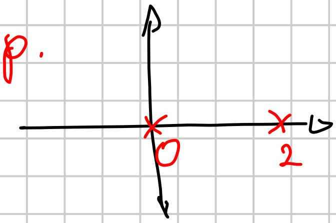
S → Punkt
→ Extremstelle
→ Wendestelle



W → P/
→ Wendestelle

Bsp. 2.9. $f(x) = ax^4 + \cancel{bx^3} + cx^2 + \cancel{dx^1} + e$

- symm. zur y-Achse ⇒ gerade Exp.
- 0, 2 Schnittstellen x-Achse
  → Nullstellen (0|0) (2|0)
- P(4|12)



$$\text{I}: f(0) = 0 = \cancel{a \cdot 0^4} + \cancel{c \cdot 0^2} + e \Rightarrow e = 0$$

$$\text{II}: f(2) = 0 = a \cdot 2^4 + c \cdot 2^2 + \cancel{e}^{\,0}$$

$$\text{III}: f(4) = 12 = a \cdot 4^4 + c \cdot 4^2 + \cancel{e}^{\,0}$$

$$\text{II}: 0 = 16 \cdot a + 4c \Rightarrow 4c = -16a$$

$$\boxed{c = -4a}$$

$$\text{III}: 12 = 256 \cdot a + 16c$$

$$12 = 256 \cdot a + 16 \cdot (-4a)$$

$$12 = 192a \quad | :192$$

$$\frac{12}{192} = \underline{\underline{a = \frac{1}{16}}}$$

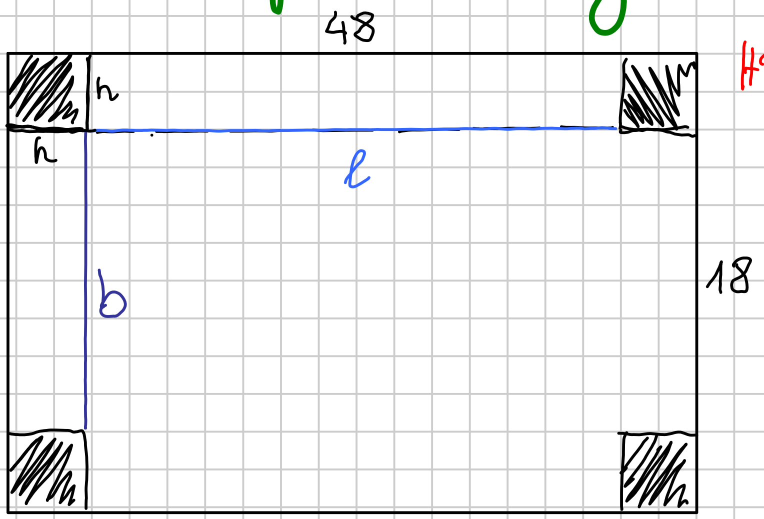
▸ Frac

$$\Rightarrow c = -4 \cdot \frac{1}{16} = -\frac{1}{4}$$

$$\underline{\underline{f(x) = +\frac{1}{16}x^4 - \frac{1}{4}x^2}}$$

Extremwertaufgaben

↳ Optimierung



Hauptbedingung

HB: $V = \ell \cdot b \cdot h \rightarrow MAX$

$V(\ell, b, h) = \ell \cdot b \cdot h$

Nebenbedingung(en)

$\ell = 48 - 2h$ $\qquad \ell \sim h$

$b = 18 - 2h$ $\qquad b \sim h$

$\Rightarrow$ HB: $V(h) = (48-2h)\cdot(18-2h)\cdot h$

$V(h) = (48-2h)(18h - 2h^2)$

$V(h) = 864h - 36h^2 - 96h^2 + 4h^3$

Zielfunktion: $V(h) = 4h^3 - 132h^2 + 864h$

$\Rightarrow$ Ableitung $V'(h) = 12h^2 - 264h + 864 \overset{!}{=} 0$

$\Rightarrow \underline{h_1 = 4}$ (ABC-Formel)

$[h_2 = 18]$

sinnlos, weil b nur 18 cm !

A: Man muss 4 cm große Quadrate an den Ecken herausschneiden um den volumsgrößten, rechteckigen offenen Behälter zu bekommen

$V(4) = (48-2\cdot4)\cdot(18-2\cdot4)\cdot4 = 1600\, cm^3 = V_{max}$

Bsp. Welches Rechteck hat bei gegebenem Umfang die größte Fläche?

HB: $A(\ell, b) = \ell \cdot b$

NB: Geg.: U (Formvariable) $U = 1m$

$$U = 2(\ell + b)$$
$$\frac{U}{2} = \ell + b$$
$$\frac{U}{2} - \ell = b \qquad \ell \sim b$$

$\Rightarrow$ HB: $A(\ell) = \ell \cdot \left(\frac{U}{2} - \ell\right)$

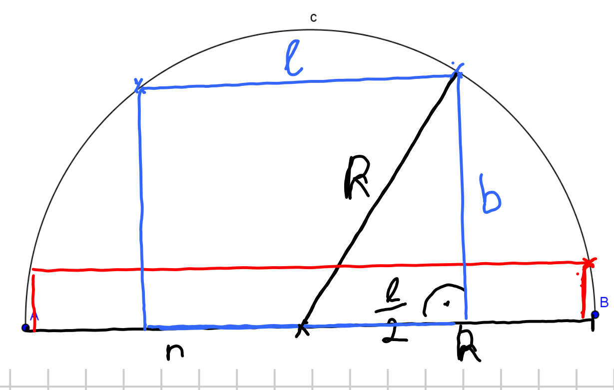
ZF: $A(\ell) = \frac{U}{2} \cdot \ell - \ell^2$

$$A'(\ell) = \frac{U}{2} - 2\ell \overset{!}{=} 0$$
$$\frac{U}{2} = 2\ell \quad | :2$$
$$\frac{U}{4} = \ell$$
$$\Rightarrow b = \frac{U}{2} - \frac{U}{4} = \frac{U}{4}$$

$$\ell = b = \frac{U}{4} \qquad A = \frac{U}{4} \cdot \frac{U}{4} = \frac{U^2}{16}$$

A: Das Rechteck mit der größten Fläche bei geg. Umfang ist das Quadrat mit $\ell = b = \frac{U}{4}$ und $A = \frac{U^2}{16}$

B.2.14



HB: $U(\ell, b) = 2(\ell + b)$

NB: $R^2 = b^2 + \left(\frac{\ell}{2}\right)^2$

$$\ell \sim b$$
$$b^2 = R^2 - \left(\frac{\ell}{2}\right)^2$$

$$b = \sqrt{R^2 - \left(\frac{\ell}{2}\right)^2}$$

$\Rightarrow$ HB:

$$U(\ell) = 2\cdot\left(\ell + \sqrt{R^2 - \left(\frac{\ell}{2}\right)^2}\right) = 2\left(\ell + \left(R^2 - \frac{\ell^2}{4}\right)^{\frac{1}{2}}\right)$$

$$U'(\ell) = \cancel{2}\cdot\left(1 + \frac{1}{2}\cdot\left(R^2 - \left(\frac{\ell}{2}\right)^2\right)^{-\frac{1}{2}}\cdot\left(0 - \frac{\cancel{2}^{1}\ell}{\cancel{4}_{2}}\right)\right) \stackrel{!}{=} 0$$

$$1 + \frac{1}{2}\cdot\frac{1}{\sqrt{R^2 - \left(\frac{\ell}{2}\right)^2}}\cdot\left(-\frac{\ell}{2}\right) = 0 \quad \Big| -1$$

$$\overset{+}{-}\,\frac{\ell}{4\cdot\sqrt{R^2 - \left(\frac{\ell}{2}\right)^2}} = \overset{+}{-}1$$

$$\ell = 4\cdot\sqrt{R^2 - \left(\frac{\ell}{2}\right)^2} \quad \Big/\,^2$$

$$\ell^2 = 16\cdot\left(R^2 - \left(\frac{\ell}{2}\right)^2\right)$$

$$\ell^2 = 16R^2 - 16\cdot\left(\frac{\ell}{2}\right)^2$$

$$\ell^2 = 16R^2 - \cancel{16}^{4}\cdot\frac{\ell^2}{\cancel{4}}$$

$$5\ell^2 = 16R^2$$

$$\ell^2 = \frac{16R^2}{5}$$

$$\ell = \sqrt{\frac{16R^2}{5}} = \boxed{\frac{4R}{\sqrt{5}}}$$

$$b = \sqrt{R^2 - \frac{\ell^2}{4}} = \sqrt{R^2 - \frac{\frac{16R^2}{5}}{4}} = \sqrt{\frac{20}{20}R^2 - \frac{16R^2}{20}} =$$

$$= \sqrt{\frac{\cancel{4}^{1}R^2}{\cancel{20}_{5}}} = \boxed{\frac{R}{\sqrt{5}}}$$

$$u = 2(\ell + b) = 2\cdot\left(\frac{4R}{\sqrt{5}} + \frac{R}{\sqrt{5}}\right) = \underline{\frac{10R}{\sqrt{5}}}$$

$$\ell = 4\cdot b$$