Differenzialrechnung - Extremwertaufgaben

Zum Aufwärmen

Lisa zieht folgende fünf Ziffernkarten:

7	0	5	8	2

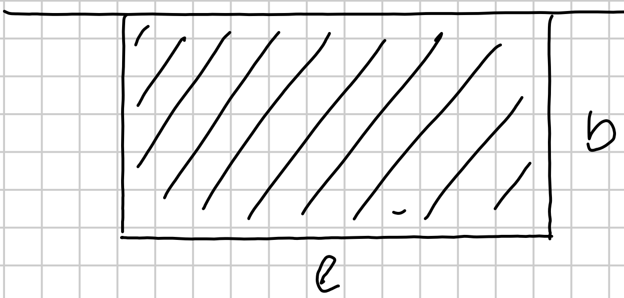
Sie legt mit vier dieser Ziffernkarten zwei zweistellige Zahlen. Dann multipliziert sie diese Zahlen miteinander. Das Ergebnis ist ungefähr 1 600.

Welche Zahlen könnte sie gelegt haben? Schreib deine Rechnung auf.

$78 \cdot 20 = 1560$

$57 \cdot 28 \approx 1596$

B.2.13 59/39



(2) $A \rightarrow MAX$

$A(\ell, b) = \ell \cdot b$

(3) $u = 100m = 2b + \ell$

$\ell \sim b$

$\ell = 100 - 2b$

(4) NB → HB

$A(b) = (100 - 2b) \cdot b =$

$A(b) = 100 \cdot b - 2b^2$ ZF (Zielfunktion)

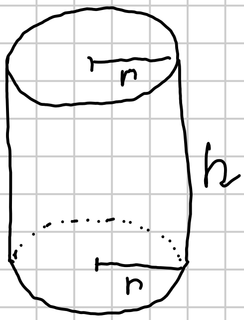
(5) $A'(b) = 100 - 4b \overset{!}{=} 0$ $[A''(b) = -4 < 0]$ MAX

$b = 25\,m$

(6) $\Rightarrow \ell = 100 - 2 \cdot 25 = 50\,m$

(7) $A_{max} = l \cdot b = 1250\,m^2$

Bsp. Wie muss man ein zylindrisches Gefäß dimensionieren damit bei 1l Fassungsvermögen möglichst wenig Material zum Bau verwendet wird.



HB: $O = 2G + M \to MIN$

$O(r,h) = 2 \cdot r^2\pi + 2r\pi \cdot h$

Wasser

NB: $\boxed{V} = 1l \;\hat{=}\; 1\,dm^3 \;\hat{=}\; 1kg$

$V = G \cdot h$

$V = r^2\pi \cdot h$ $\qquad r \sim h$

$$\boxed{\frac{V}{r^2\pi} = h}$$

$\Rightarrow$ $O(r) = 2r^2\pi + 2\cancel{r}\cancel{\pi} \cdot \frac{V}{r^{\cancel{2}}\cancel{\pi}}$ $\qquad$ Ansatzvereinfachung

$O(r) = 2r^2\pi + \frac{2V}{r} = \cancel{2} \cdot \left(r^2\pi + \frac{V}{r}\right)$ $\qquad$ ZF

$V \cdot r^{-1}$

$O'(r) = \cancel{2} \cdot \left(2r\pi + V \cdot (-1) \cdot r^{-2}\right)$

$= \cancel{2} \cdot \left(2r\pi - \frac{V}{r^2}\right) \overset{!}{=} 0 \quad | :2$

$2r\pi = \frac{V}{r^2} \quad | \cdot r^2$

$2r^3\pi = V$

$r^3 = \frac{V}{2\pi}$

$V = 1$

$\boxed{r = \sqrt[3]{\frac{V}{2\pi}}} = 0{,}5419\,dm = 5{,}42\,cm$

$r^2 = \left[\left(\frac{V}{2\pi}\right)^{\frac{1}{3}}\right]^2$

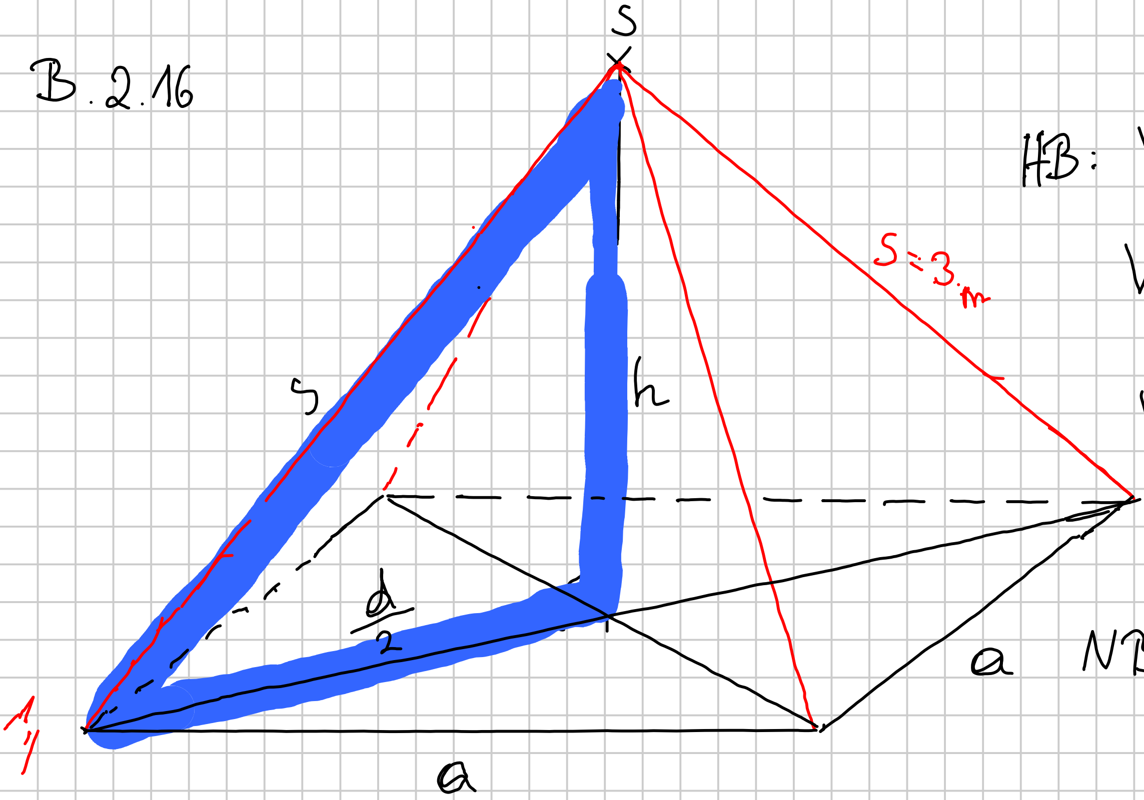
$$h = \frac{V}{r^2\pi} = \frac{V}{\left(\frac{V}{2\pi}\right)^{\frac{2}{3}} \cdot \pi} = \frac{V^{\frac{1}{3}}}{\frac{\cancel{V^{\frac{2}{3}}}}{2^{\frac{2}{3}}\cancel{\pi^{\frac{2}{3}}}} \cdot \pi^{\frac{1}{3}}} = \frac{V^{\frac{1}{3}} \cdot 2^{\frac{2}{3}}}{\pi^{\frac{1}{3}}} =$$

$$= \left(\frac{V^1 \cdot 2^2}{\pi^1}\right)^{\frac{1}{3}} = \sqrt[3]{\frac{2 \cdot 4V}{2 \cdot \pi}} = \sqrt[3]{\frac{8V}{2\pi}} = 2 \cdot \sqrt[3]{\frac{V}{2\pi}}$$

$$\Rightarrow h = 2 \cdot r = d$$

$$\underline{\underline{h = 10{,}84\ \text{cm}}} \qquad ((O_{min}))$$

B.2.16

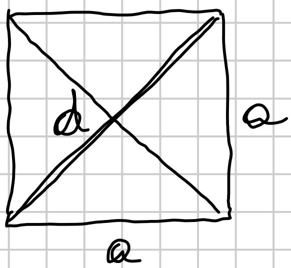


HB: $V \Rightarrow$ MAX

$$V = \frac{G \cdot h}{3}$$

$$V(a,h) = \frac{a^2 \cdot h}{3}$$

NB:



$$\frac{d}{2} = \frac{\sqrt{a^2+a^2}}{2} = \frac{\sqrt{2a^2}}{2} = \frac{a \cdot \sqrt{2}}{2}$$

$$s^2 = \left(\frac{d}{2}\right)^2 + h^2$$

$$s^2 = \left(\frac{a \cdot \sqrt{2}}{2}\right)^2 + h^2$$

$$s^2 = \frac{a^2 \cdot \cancel{2}}{\cancel{4}\,2} + h^2 = \frac{a^2}{2} + h^2 \qquad a \sim h$$

$$s^2 - h^2 = \frac{a^2}{2} \quad | \cdot 2$$

$$2(s^2 - h^2) = a^2$$

$\Rightarrow$ NB $\rightarrow$ HB

$$V(h) = \frac{2(s^2-h^2)\cdot h}{3}$$

$$\overline{V}(h) = \cancel{\tfrac{2}{3}} \cdot (s^2 h - h^3)$$

$\frac{1}{4} \cdot 3 = \frac{3}{4}$

$\frac{1}{4} \cdot 3.5 = \frac{3.5}{4}$

Ansatz vereinf.

$3^2 \qquad 3^2 \cdot h$

$$\overline{V}'(\underline{h}) = s^2 - 3h^2 \stackrel{!}{=} 0$$

$$s^2 = 3h^2 \quad | :3$$

$$\frac{s^2}{3} = h^2$$

$$\sqrt{\frac{s^2}{3}} = h$$

$$\frac{s}{\sqrt{3}} = h \stackrel{s=3}{=} 1{,}73\,m$$

$$\Rightarrow a^2 = 2\cdot(s^2 - h^2) = 2\cdot\left(s^2 - \frac{s^2}{3}\right)$$

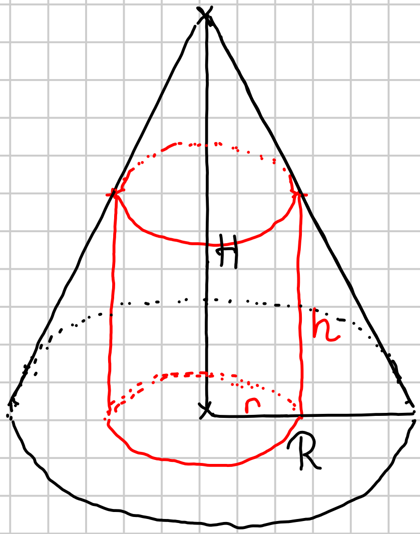
$$a^2 = 2 \cdot \frac{2s^2}{3} = \frac{4s^2}{3}$$

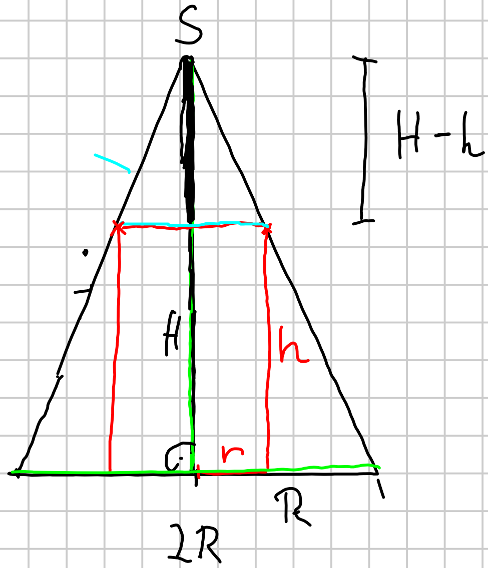
$$\Rightarrow a = \sqrt{\frac{4s^2}{3}} = \frac{2\cdot s}{\sqrt{3}}$$

$\underline{\underline{a = 2h}}$

$a = 3{,}46\,m$

Bsp. 2 18





$V_z(r,h) = r^2\pi\cdot h$ $\quad$ NB $\quad H : 2R = H-h : 2r$

$r \sim h$

$$\frac{H}{\cancel{2}R} = \frac{H-h}{\cancel{2}r}$$

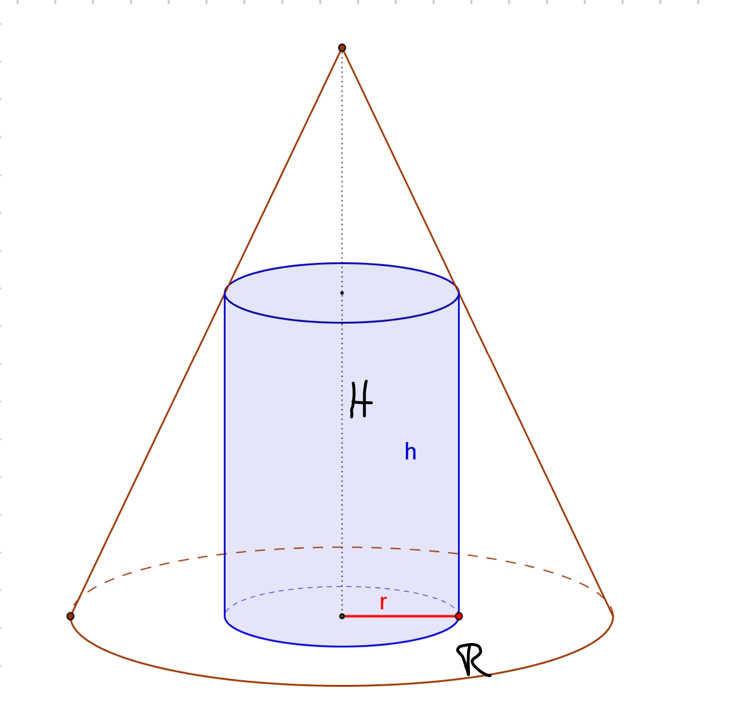
$$H\cdot r = (H-h)\cdot R$$

$$\frac{H\cdot r}{R} = H-h \quad / +h / -$$

$$h = H - \frac{H\cdot r}{R}$$

$$\underline{h = H\left(1-\frac{r}{R}\right)}$$

$\Rightarrow$ HB



$$V(r) = r^2\cdot\cancel{\pi}\cdot\cancel{H}\left(1-\frac{r}{R}\right)$$

$$\bar{V}(r) = r^2 - \frac{r^3}{R}$$

$$\bar{V}'(r) = 2r - \frac{3r^2}{R} = 0$$

$$r\cdot\left(2-\frac{3r}{R}\right) = 0$$

$r=0$ $\quad 2-\frac{3r}{R}=0$

$$2r = \frac{3r^2}{R} \quad / :r$$

$$2 = \frac{3r}{R} \quad / \cdot R / :3$$

$$\underline{\underline{\frac{2R}{3} = r}}$$

$$h = H\cdot\left(1-\frac{\frac{2R}{3}}{R}\right) = H\cdot\left(1-\frac{2\cancel{R}}{3\cancel{R}}\right) = \frac{H}{3}$$

$$V_{max} = r^2\cdot\pi\cdot h = \left(\frac{2R}{3}\right)^2\cdot\pi\cdot\frac{H}{3} = \frac{4R^2\pi\cdot H}{27}$$

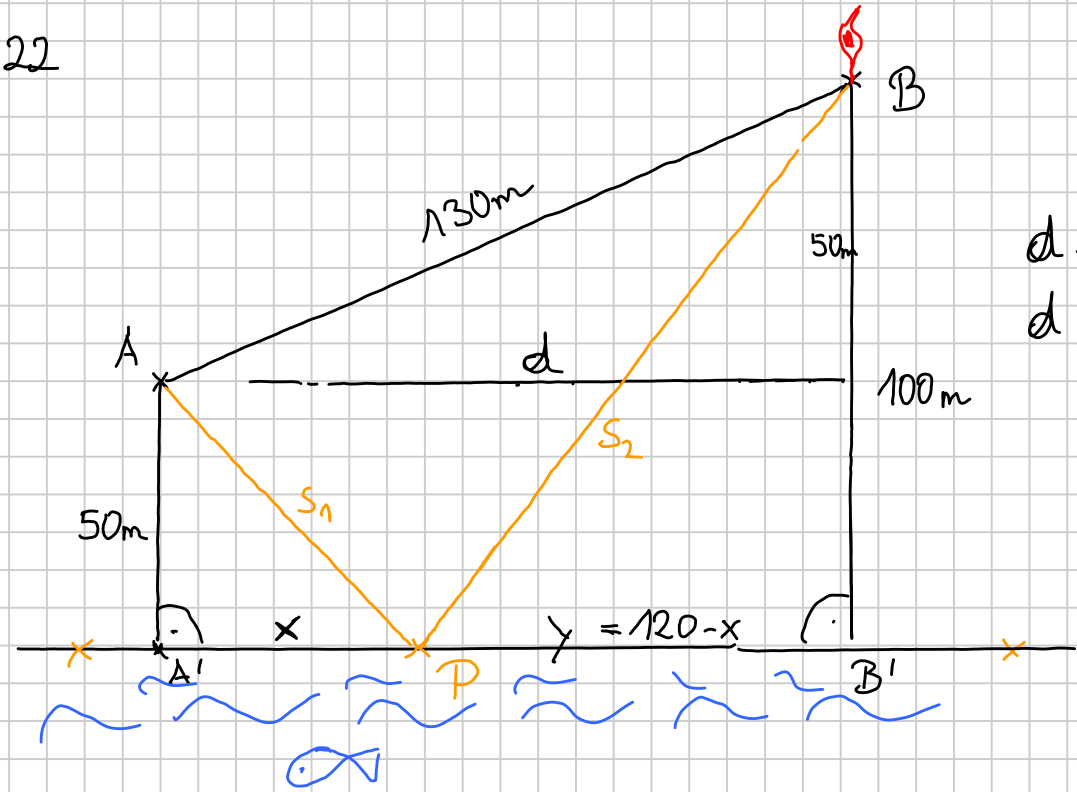
$$V_K = \frac{R^2\pi\cdot H}{3}$$

$$\frac{\cancel{R^2}\cancel{\pi}\cdot\cancel{H}}{\cancel{3}} : \frac{4\cancel{R^2}\cancel{\pi}\cdot\cancel{H}}{\cancel{27}\,9}$$

$1 : \frac{4}{9}$

$9 : 4$

Bsp. 2.22



$d = \sqrt{130^2 - 50^2}$

$d = 120$

HB: $s = s_1 + s_2 \rightarrow$ MIN $\qquad t = \frac{s_1}{v_1} + \frac{s_2}{v_2}$

NB $\quad s_1 = \sqrt{x^2 + 50^2} \qquad s_1 \sim x$

$s_2 = \sqrt{100^2 + (120 - x)^2} \qquad s_2 \sim x$

$$s(x) = \sqrt{x^2 + 50^2} + \sqrt{100^2 + (120 - x)^2}$$

$$s(x) = \sqrt{x^2 + 2500} + \sqrt{10.000 + 14400 - 240x + x^2}$$

$$s(x) = \sqrt{x^2 + 2500} + \sqrt{x^2 - 240x + 24.400}$$

$$\sqrt{f(x)}' = \frac{f'(x)}{2 \cdot \sqrt{f(x)}}$$

$$s' = \frac{\cancel{2} \cdot x}{\cancel{2} \cdot \sqrt{x^2 + 2500}} + \frac{\overset{\cancel{2}(x - 120)}{2x - 240}}{\cancel{2} \cdot \sqrt{x^2 - 240x + 24.400}} \overset{!}{=} 0$$

$$\frac{x}{\sqrt{x^2+2500}} = - \frac{(x-120)}{\sqrt{x^2-240x+24.400}} \quad \Big| \; ^{②}$$

$$\frac{x^2}{x^2+2500} = \frac{x^2-240x+14.400}{x^2-240x+24.400}$$

$$x^2 \cdot (x^2-240x+24.400) = (x^2+2500)\cdot(x^2-240x+14.400)$$

$$\cancel{x^4} - \cancel{240x^3} + 24.400x^2 = \cancel{x^4} - \cancel{240x^3} + 14.400x^2 + 2500x^2 - 600.000x$$

$$36.000.000$$

$$7500x^2 + 600.000x - 36.000.000 = 0$$

ABC-Formel

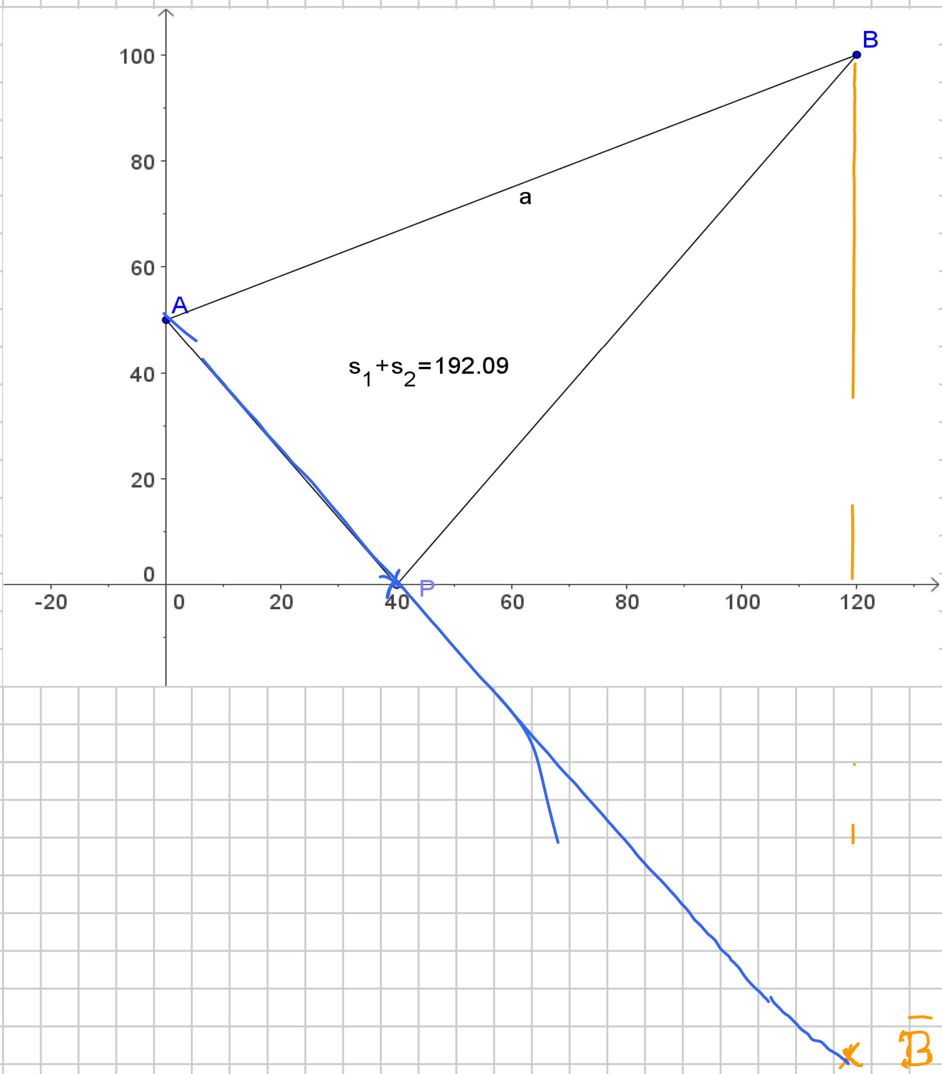
$A = 75$

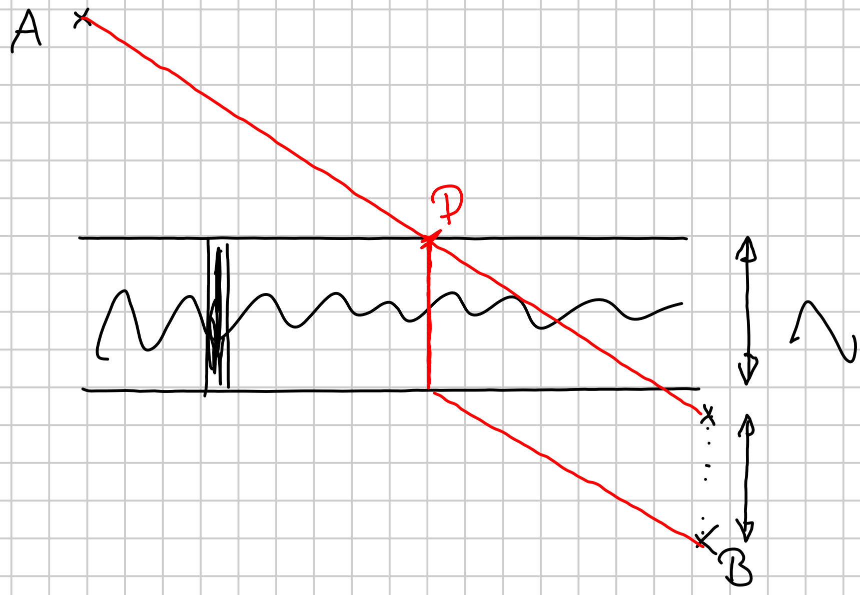
$B = 6000$

$C = -360000$

sinnlos $(x_1 = -120)$

$x_2 = 40 \text{ m}$ von A' weg





Umkehraufgaben

(0|0)

245 Eine durch den Koordinatenursprung gehende Polynomfunktion dritten Grads geht durch den Punkt $P_2(2 \mid 2)$ und berührt die x-Achse bei $x_3 = 4$.

$f(x) = ax^3 + bx^2 + cx + d$

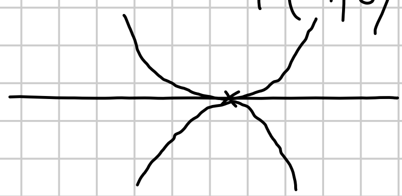
$f'(x) = 3ax^2 + 2bx + c$

$f''(x) = 6ax + 2b$

$P(2|2)$

$x_3 = 4 \Rightarrow y_3 = 0$

$P(4|0) = E$



I: $f(2) = 2 = a \cdot 8 + b \cdot 4 + c \cdot 2$

II: $f(4) = 0 = a \cdot 64 + 16b + c \cdot 4$

III: $f'(4) = 0 = 3a \cdot 16 + 2b \cdot 4 + c$

IV: $f(0) = 0 = a \cdot 0^3 + b \cdot 0^2 + c \cdot 0^1 + d \Rightarrow d = 0$

$A = \begin{pmatrix} 8 & 4 & 2 \\ 64 & 16 & 4 \\ 48 & 8 & 1 \end{pmatrix}$ $b = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$ $A^{-1} * b$

$a = 0{,}25$ $b = -2$ $c = 4$ $d = 0$

$f(x) = \frac{1}{4}x^3 - 2x^2 + 4x$

247 Eine Polynomfunktion vierten Grads, die die x-Achse im Ursprung berührt, hat im Punkt $P_2(2 \mid 2)$ eine gegen die positive x-Achse unter einem Winkel $\alpha = 45°$ geneigte Wendetangente.

$f(x) = ax^4 + bx^3 + cx^2 + dx + e$

$f'(x) = 4ax^3 + 3bx^2 + 2cx + d$

$f''(x) = 12ax^2 + 6bx + 2c$

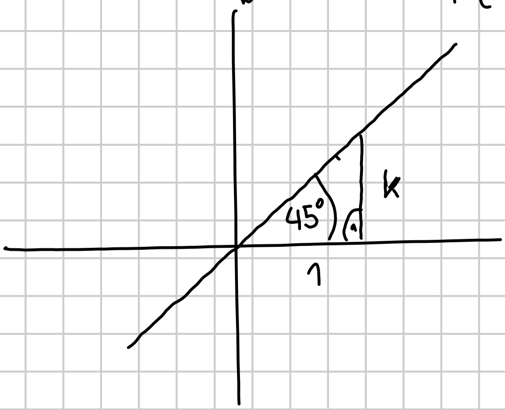
I: $P(2|2) \in f(x)$: $f(2) = 2 = 16a + 8b + 4c + 2d + e$

II: $(0|0) \in f(x)$: $f(0) = 0 = e$

III: $x=0$ E : $f'(0) = 0 = d$

IV: $x=2$ W : $f''(2) = 0 = 48a + 12b + 2c$

V: α 45° $\Rightarrow k_w = 1$ $f'(2) = 1 = 32a + 12b + 4c$



$\tan 45° = \frac{k}{1}$

$1 = k$

$\boxed{\tan\alpha = k}$

$A = \begin{pmatrix} 16 & 8 & 4 \\ 48 & 12 & 2 \\ 32 & 12 & 4 \end{pmatrix}$ $b = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$